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DEIM-BASED PGD FOR PARAMETRIC NONLINEAR MODEL ORDER REDUCTION

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Abstract. A new technique for efficiently solving parametric nonlinear reduced order models in the Proper Generalized Decomposition (PGD) framework is presented here. This technique is based on the Discrete Empirical Interpolation Method (DEIM)[1], and thus the nonlinear term is interpolated using the reduced basis instead of being fully evaluated. The DEIM has already been demonstrated to provide satisfactory results in terms of computational complexity decrease when combined with the Proper Orthogonal Decomposition (POD). However, in the POD case the reduced basis is *a posteriori* known as it comes from several pre-computed snapshots. On the contrary, the PGD is an *a priori* model reduction method. This makes the DEIM-PGD coupling rather delicate, because different choices are possible as it is analyzed in this work.

1 INTRODUCTION

The efficient resolution of complex models (in the dimensionality sense) is probably the essential objective of any model reduction method. This objective has been clearly reached for many linear models encountered in physics and engineering [2, 3]. However, model order reduction of nonlinear models, and specially, of parametric nonlinear models, remains as an open issue. Using classic linearization techniques such Newton method, both the nonlinear term and its Jacobian must be evaluated at a cost that still depends on the dimension of the non-reduced model [1]. The Discrete Empirical Interpolation Method (DEIM), which the discrete version of the Empirical Interpolation Method (EIM) [4], proposes to overcome this difficulty by using the reduced basis to interpolate the nonlinear term. The DEIM has been used with Proper Orthogonal Decomposition (POD) [5, 1] where the reduced basis is *a priori* known as it comes from several pre-computed snapshots. In this work, we propose to use the DEIM in the Proper Generalized Decomposition (PGD) framework [5, 6], which is an *a priori* model reduction technique, and thus the nonlinear term is interpolated using the reduced basis that is being constructed during the resolution.

2 DEIM-based PGD FOR NONLINEAR MODELS

Consider a certain model in the general form:

$$\mathcal{L}(u) + \mathcal{F}_{NL}(u) = 0 \tag{1}$$

where $\mathcal{L}$ is a linear differential operator and $\mathcal{F}_{NL}$ is a nonlinear function, both applying over the unknown $u(\boldsymbol{x}), \boldsymbol{x} \in \Omega = \Omega_1 \times \ldots \times \Omega_d \subset \mathbb{R}^d$, which belongs to the appropriate functional space and respects some boundary and/or initial conditions. Using the PGD method implies constructing a basis $\mathcal{B} = \{\phi_1, \ldots, \phi_N\}$ such that the solution can be written as:

$$u(\boldsymbol{x}) \approx \sum_{i=1}^{N} \alpha_i \cdot \phi_i(\boldsymbol{x})$$

where α_i are coefficients, and

$$\phi_i(\boldsymbol{x}) = P_{i1}(x_1) \cdot \ldots \cdot P_{id}(x_d) \ , \ i = 1, \ldots, N$$

being $P_{ij}(x_j), j = 1, \ldots, d$, functions of a certain coordinate $x_j \in \Omega_j$. In the linear case, the basis $\mathcal{B}$ can be constructed sequentially by solving a nonlinear problem at each step in order to find functions P_{ij}. In the nonlinear case a linearization scheme for Eq. (1) is compulsory, but evaluating the nonlinear term is still as costly as in the non-reduced model. The DEIM method proposes to circumvent this inconvenient by performing an interpolation of the nonlinear term using the basis functions. In a POD framework, these functions come from the precomputed snapshots, but in a PGD framework these functions are constructed by using the PGD algorithm. Here we propose to proceed as follows:

1. Solve the linear problem: find u^0 such that $\mathcal{L}(u^0) = 0 \rightarrow \mathcal{B}^0 = \{\phi_1^0, \ldots, \phi_{N_0}^0\}$

2. Select a set of points $\mathcal{X}^0 = \{\boldsymbol{x}_1^0, \ldots, \boldsymbol{x}_{N_0}^0\}$. Later on we explain how to make an appropriate choice.

3. Interpolate the nonlinear term $\mathcal{F}_{NL}$ using functions $\mathcal{B}^0$ in the points $\mathcal{X}^0$. Or in other words, find the coefficients φ_i^0 such as:

$$\mathcal{F}_{NL}\left(u_m^0\right) \equiv \mathcal{F}_{NL}\left(u^0(\boldsymbol{x}_m^0)\right) = \sum_{i=1}^{N_0} \varphi_i^0 \cdot \phi_i^0(\boldsymbol{x}_m^0) \ , \ m = 1, \ldots, N_0$$

The previous equation represents a linear system which will be invertible if $\mathcal{B}^0$ is linearly independent (and it is because it comes from the solution of the linear problem) and all points of $\mathcal{X}^0$ are different.

4. Once we have computed $\{\varphi_1^0, \ldots, \varphi_{N_0}^0\}$, the interpolation of the nonlinear term reads:

$$\mathcal{F}_{NL}(u) \approx b_0 = -\sum_{i=1}^{N_0} \varphi_i^0 \cdot \phi_i^0$$

   And therefore, the linearized problem writes:

$$\mathcal{L}(u) = b_0 \tag{2}$$

5. At this point, three options can be thought:

   (a) Restart the separated representation, i.e., find u^1 such that:

$$\mathcal{L}(u^1) - b_0 = 0$$

   Applying the PGD method we will obtain a new reduced basis $\mathcal{B}^1 = \{\phi_1^1, \ldots, \phi_{N_1}^1\}$.

   (b) Reuse the solution u^0, i.e. $u^1 = u^0 + \widetilde{u}$. Then, we seek $\widetilde{u}$ such that:

$$\mathcal{L}(\widetilde{u}) = b_0 - \mathcal{L}(u^0)$$

   We solve this problem by applying the PGD method, i.e. $\widetilde{\mathcal{B}} = \{\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{\widetilde{N}}\}$ and then $\mathcal{B}^1 = \mathcal{B}^0 \oplus \widetilde{\mathcal{B}}$ and $N_1 = N_0 + \widetilde{N}$.

   (c) Reuse by projecting. In this case we consider

$$u^{0,1}(\boldsymbol{x}) = \sum_{i=1}^{N_0} \eta_i^0 \cdot \phi_i^0(\boldsymbol{x})$$

   which introduced into Eq. (2) allows computing coefficients η_i^0. Then the approximation is enriched by considering $u^1 = u^{0,1} + \widetilde{u}$. In this case, we seek $\widetilde{u}$ such that:

$$\mathcal{L}(\widetilde{u}) = b_0 - \mathcal{L}(u^{0,1})$$

   Once this problem is solved by applying the PGD method, i.e. $\widetilde{\mathcal{B}} = \{\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{\widetilde{N}}\}$ and then $\mathcal{B}^1 = \mathcal{B}^0 \oplus \widetilde{\mathcal{B}}$ and $N_1 = N_0 + \widetilde{N}$.

6. From this point we repeat the precedent steps: let us assume that we have already computed u^k. Then select a set of points $\mathcal{X}^k = \{\boldsymbol{x}_1^k, \ldots, \boldsymbol{x}_{N_k}^k\}$, interpolate the nonlinear term using $\mathcal{B}^k$, and find u^{k+1}, until a certain convergence criteria is reached.

3 ELECTION OF THE INTERPOLATION POINTS

An open question is how to choose the points $\mathcal{X}^k = \{\boldsymbol{x}_1^k, \ldots, \boldsymbol{x}_{N_k}^k\}$. Consider that a certain computation step we know the reduced approximation basis:

$$\mathcal{B}^k = \{\phi_1^k, \ldots, \phi_{N_k}^k\}$$

Following [1, 4], we consider:

$$\boldsymbol{x}_1^k = \arg\max_{\boldsymbol{x}\in\Omega} |\phi_1^k(\boldsymbol{x})|$$

Then we compute c_1 from

$$c_1 \cdot \phi_1^k(\boldsymbol{x}_1^k) = \phi_2^k(\boldsymbol{x}_1^k)$$

which allows defining:

$$r_2(\boldsymbol{x}) = \phi_2^k(\boldsymbol{x}) - c_1 \cdot \phi_1^k(\boldsymbol{x})$$

from where we can compute the following point, $\boldsymbol{x}_2^k$ as:

$$\boldsymbol{x}_2^k = \arg\max_{\boldsymbol{x}\in\Omega} |r_2(\boldsymbol{x})|$$

As by construction $r_2(\boldsymbol{x}_1^k) = 0$, we can ensure that $\boldsymbol{x}_1^k \neq \boldsymbol{x}_2^k$. This process can be generalized and thus, if we are looking for $\boldsymbol{x}_j^k, j \leq k$, the following function can be constructed:

$$r_j(\boldsymbol{x}) = \phi_j^k(\boldsymbol{x}) - \sum_{i=1}^{j-1} c_i \cdot \phi_i^k(\boldsymbol{x})$$

where coefficients $c_i, 1 \leq i \leq j-1$, need to be computed. It can be done by imposing that:

$$r_j(\boldsymbol{x}_l^k) = 0 = \phi_j^k(\boldsymbol{x}_l^k) - \sum_{i=1}^{j-1} c_i \cdot \phi_i^k(\boldsymbol{x}_l^k) \ , \ \ l = 1, \ldots, j-1$$

that constitutes a linear system whose solution results the coefficients c_i. And then we compute the sought point:

$$\boldsymbol{x}_j^k = \arg\max_{\boldsymbol{x}\in\Omega} |r_j(\boldsymbol{x})| \tag{3}$$

It must be pointed out that, in principle, Eq. (3) implies reconstructing the solution, that is to say, to compute explicitly the functions $\phi_l^k, l = 2, \ldots, j-1$ from the separated functions $P_{l,s}^k(x_s)$ with $s = 1, \ldots, d$. For $l = 1$, notice that things are much simpler:

$$\boldsymbol{x}_1^k = \left(x_{1,1}^k, \ldots, x_{1,d}^k\right)$$

with

$$x_{1,s}^k = \arg\max_{x_s\in\Omega_s} |P_{1,s}^k(x_s)| \ , \ \ s = 1, \ldots, d$$

For $l > 1$ some simplifying procedures must be defined for avoiding the solution reconstruction and improve the performance in the multi-parametric case. The analysis of such procedures constitutes a work in progress.

4 NUMERICAL EXAMPLE

Aiming to prove the ability of the DEIM-based PGD method for solving nonlinear problems, we consider the transient heat equation with a quadratic nonlinearity:

$$\frac{\partial u}{\partial t} - \Delta u + u^2 = 0 \ , \ (\boldsymbol{x}, t) \in \Omega \times (0, T] \tag{4}$$

being $\Omega = [0,1]\times[0,1] \subset \mathbb{R}^2$. The initial condition reads $u(\boldsymbol{x}, t = 0) = 0$ and the boundary conditions are given by $u(x = 0, y = 0, t) = u(x = 1, y = 0, t) = 0$ and $\nabla u \cdot \boldsymbol{n}\,(x = 0.5, y = 1, t) = 1$. Outside these boundaries, a zero-flux condition is considered.

A space-time separated representation is sought in this case:

$$u(\boldsymbol{x}, t) = \sum_{i=1}^{N} X_i(\boldsymbol{x}) \cdot T_i(t) \tag{5}$$

We use here the reuse option, that is to say, the reduced basis is enriched without projection. Using the notation introduced in the previous section the convergence was reached after the construction of $k = 4$ reduced bases in which the nonlinear term was interpolated, for a relative error less than 1% to the reference solution. However, a relative error of 0.5% cannot be attained in spite of the number of basis enrichment. The final solution involved $N = 52$ functional products $\phi_i = X_i(\boldsymbol{x}) \cdot T_i(t)$. Fig. 1 compares the time evolution at different locations obtained with the DEIM based PGD and the exact solution. Then Fig. 2 and 3 depict the four fist space and time modes respectively. From these results we can conclude on the potentiality of the proposed technology for solving non-linear eventually multi-parametric models.

5 CONCLUSIONS

This work presents the DEIM-based PGD technique for solving efficiently reduced nonlinear models. The improvement is achieved by interpolating the nonlinear term using the reduced basis, computed as usual with the PGD method, instead of performing its complete evaluation. As the PGD is an *a priori* model reduction technique, a progressive reduced basis enrichment must be considered, and thus up to three different choices can be done: restart the reduced basis, reuse the previous reduced basis by enrichment and reuse the reduced basis by projecting. A deep analysis of the different alternatives is in progress.

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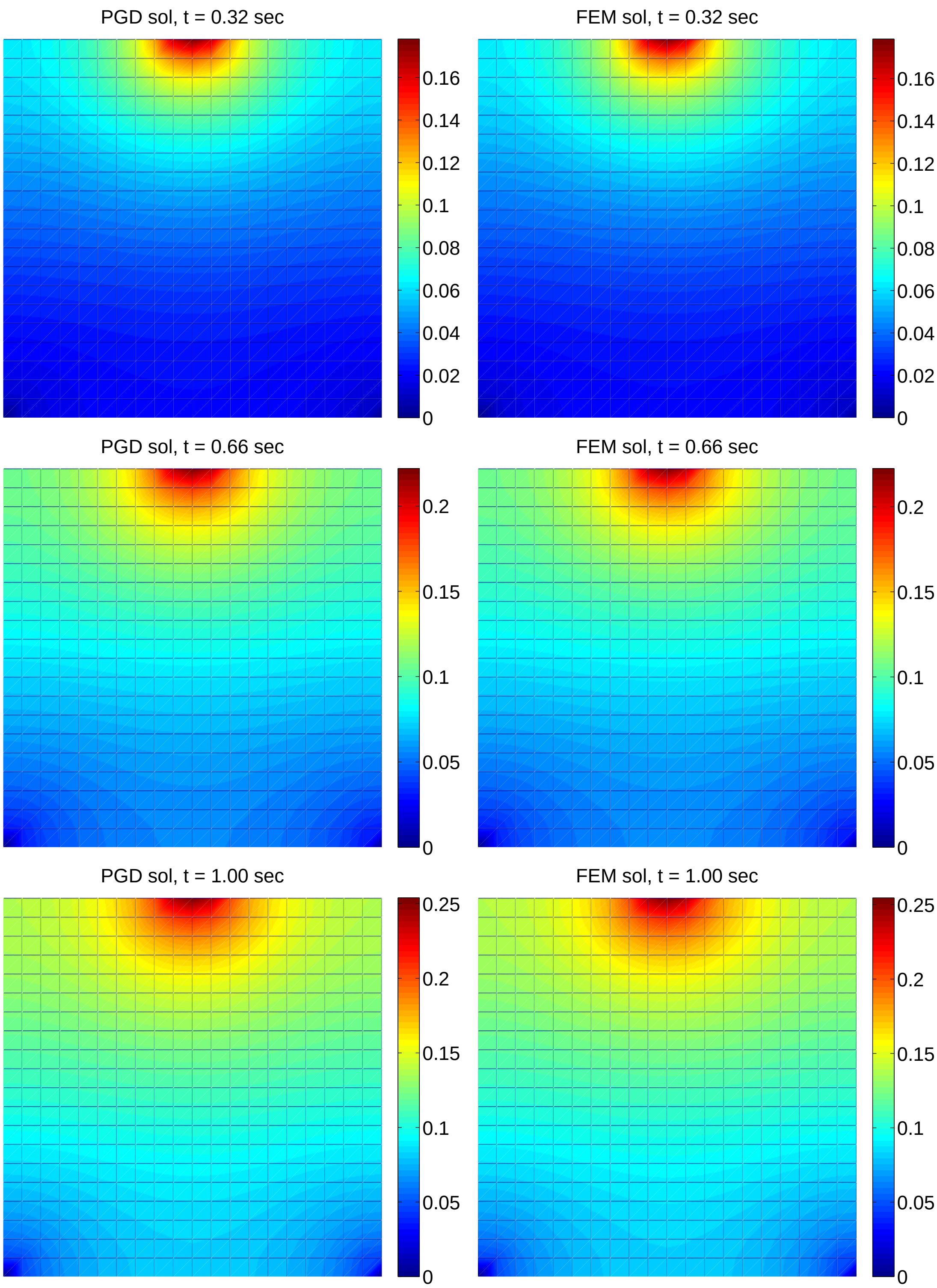

Figure 1: Comparison of the DEIM-based PGD solution (on the left) to the FEM reference solution (on the right), for three different times, $t = 0.32, 0.66, 1.00$ sec

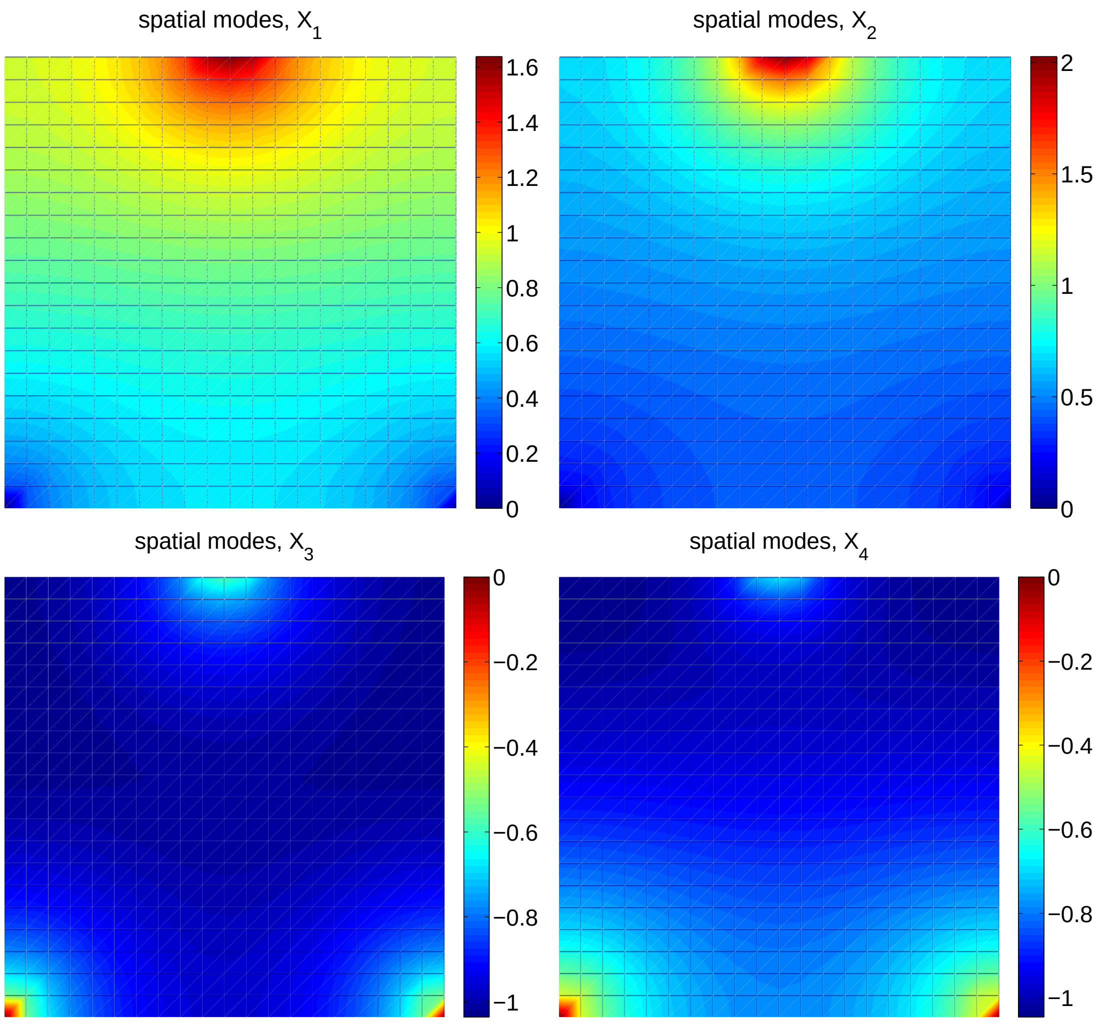

Figure 2: First four space normalized modes of the DEIM-based PGD solution

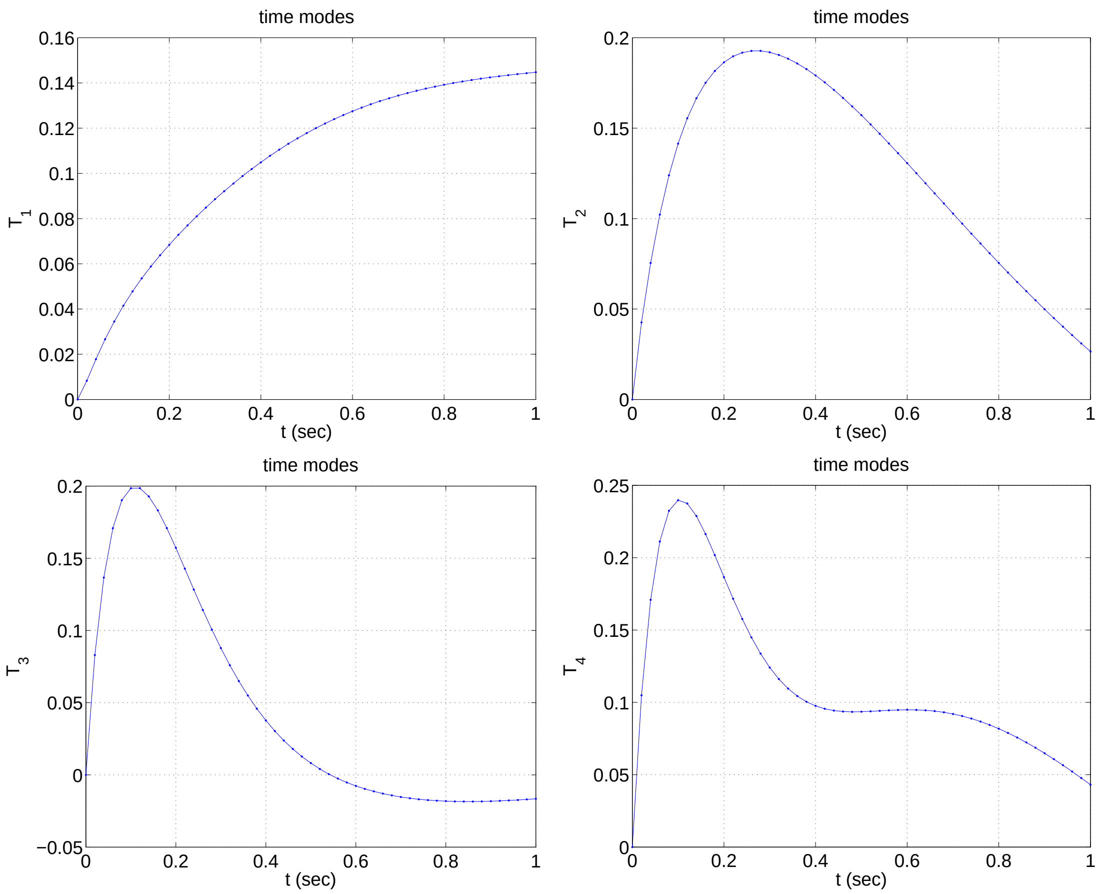

Figure 3: First four time normalized modes of the DEIM-based PGD solution