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REDUCED ORDER MULTISCALE FINITE ELEMENT METHODS BASED ON COMPONENT MODE SYNTHESIS

MATS G. LARSON*, **HÅKAN JAKOBSSON***, **AND TOR TROENG***

*Department of Mathematics and Mathematical Statistics
Umeå University
90187 Umeå, Sweden
e-mail: mats.larson@math.umu.se

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Abstract. We present a reduced order finite element method based on the variational multiscale method together with a component mode synthesis representation for the fine scale part of the solution. We derive an a posteriori error estimate in the energy norm for the discrete error in the approximation which measures the error associated with model reduction in the fine scale.

1 INTRODUCTION

In this contribution we briefly describe a recent multiscale finite element method, introduced in [6], which builds on using a reduced order model for the fine scale in a variational multiscale method, see [2] and the later developments [5].

Model reduction methods are commonly used to decrease the computational cost associated with simulations involving repeated use of large scale finite element models of for instance a complicated structure. The objective in model reduction is to find a low dimensional subspace of the finite element function space that still captures the structural behavior sufficiently well. A classical model reduction method is component mode synthesis (CMS), see [3].

In CMS the computational domain is split into subdomains and a reduced basis associated with the subdomain is constructed by solving localized constrained eigenvalue problems associated with the subdomains together with modes that represent the displacements of the interface between the subdomains, as in the Craig-Bampton method [1].

Here we construct a multiscale finite element method where the coarse scale is represented by piecewise linear continuous elements on a coarse mesh and the fine scale is defined by a CMS related approach on a refined mesh, using the coarse mesh elements as subdomains in the CMS method. The coupling modes are computed for each pair of neighboring elements and couple the response in the subdomains. Thus the fine scale is finally represented as a direct sum of functions with support in each element and functions associated with each edge supported in the two elements neighboring the edge. Adaptive reduction is accomplished by choosing a basis in each such subspace consisting of a truncated sequence of eigenmodes. The eigenmodes are numerically computed and capture fine scale effects.

We derive an a posteriori error estimate for the multiscale finite element method that can be used to automatically tune the number of subscale modes in an adaptive algorithm. For further details we refer to [6] and the previous work on a posteriori error estimates for component mode synthesis [4].

2 LINEAR ELASTICITY

The equations of linear elasticity take the form: find displacements $\boldsymbol{u}$ such that

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) + \tau \boldsymbol{u} &= \boldsymbol{f}, && \boldsymbol{x} \in \Omega, && \text{(1a)}\\
\boldsymbol{\sigma}(\boldsymbol{u}) &= 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda (\nabla \cdot \boldsymbol{u}) \boldsymbol{I}, && \boldsymbol{x} \in \Omega, && \text{(1b)}\\
\boldsymbol{u} &= \boldsymbol{0}, && \boldsymbol{x} \in \Gamma_D, && \text{(1c)}\\
\boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{g}_N, && \boldsymbol{x} \in \Gamma_N, && \text{(1d)}
\end{aligned}
$$

where $\tau \geq 0$ is a real parameter, $\boldsymbol{f}$ is a body force, $\boldsymbol{g}_N$ is a traction force, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the linear strain tensor, $\boldsymbol{\sigma}$ the stress tensor, $\boldsymbol{I}$ is the $d \times d$ identity matrix, and λ and μ are the Lamé parameters given by $\lambda = E\nu[(1+\nu)(1-2\nu)]^{-1}$ and $\mu = E[2(1+\nu)]^{-1}$, where E and ν is Young's modulus and Poisson's ratio respectively. The coefficients can have multiscale behavior, i.e. exhibit variation on a very fine scale or on multiple scales.

The corresponding variational form of (1) reads: find $\boldsymbol{u} \in V = \{\boldsymbol{v} \in [H^1(\Omega)]^d : \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0}\}$ such that

$$
A(\boldsymbol{u}, \boldsymbol{v}) = b(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V, \tag{2}
$$

where $A(\cdot,\cdot)$ is the bilinear form

$$
A(\boldsymbol{v}, \boldsymbol{w}) = a(\boldsymbol{v}, \boldsymbol{w}) + \tau(\boldsymbol{v}, \boldsymbol{w}) \tag{3}
$$

with

$$
a(\boldsymbol{v}, \boldsymbol{v}) = 2(\mu \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{w})) + (\kappa \nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{w}), \tag{4}
$$

and $b(\cdot)$ is the linear form

$$
b(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{g}_N, \boldsymbol{v})_{\Gamma_N}. \tag{5}
$$

3 MULTISCALE METHOD

Let $\mathcal{T}^H$ be a coarse mesh on Ω consisting of shape regular triangles ($d = 2$) or tetrahedra ($d = 3$) and let $\mathcal{T}^h$ be a fine mesh obtained by a sequence of uniform refinements of $\mathcal{T}^H$. Let $V^H \subset V^h$ be the corresponding spaces of continuous piecewise linear element.

We then have the following splitting

$$V^h = V^H \oplus \left(\bigoplus_{E \in \mathcal{E}^H} V_E^h \right) \oplus \left(\bigoplus_{T \in \mathcal{T}^H} V_T^h \right) \tag{6}$$

Here $V_T^h \subset V^h$ is the space of functions with support in element $T \in \mathcal{T}^H$, $\mathcal{E}^H$ is the set of edges in the coarse mesh $\mathcal{T}^H$, and if the edge E is shared by elements T_1 and T_2 in $\mathcal{T}^H$ then the edge space V_E^h is defined by

$$V_E^h = \{ v \in V^h \,:\, \mathrm{supp}(v) \subset T_1 \cup T_2, a(v,w) = 0 \;\forall w \in V_{T_1}^h \oplus V_{T_2}^h \} \tag{7}$$

To construct a basis in these subspaces we solve the following eigenvalue problems.

Basis in V_T^h: Find $(\boldsymbol{Z}, \Lambda) \in V_T^h \times \mathbb{R}^+$, such that

$$a(\boldsymbol{Z}, \boldsymbol{v}) = \Lambda(\boldsymbol{Z}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_T^h \tag{8}$$

Using modal truncation we obtain a reduced subspace $V_T^{h,m_T} \subset V_T^h$, defined by

$$V_T^{h,m_T} = \mathrm{span}\{\boldsymbol{Z}_i\}_{i=1}^{m_T}, \tag{9}$$

where $m_T \ll \dim(V_T^h)$.

Basis in V_E^h: Find $(\boldsymbol{Z}, \Lambda) \in V_E^h \times \mathbb{R}^+$, such that

$$a(\boldsymbol{Z}, \boldsymbol{v}) = \Lambda(\boldsymbol{Z}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_E^h \tag{10}$$

Using modal truncation we obtain a reduced subspace $V_E^{h,m_E} \subset V_E^h$, defined by

$$V_E^{h,m_E} = \mathrm{span}\{\boldsymbol{Z}_i\}_{i=1}^{m_E}, \tag{11}$$

where $m_E \ll \dim(V_E^h)$.

Finally, we arrive at the reduced order space

$$V^{h,\boldsymbol{m}} = V^H \oplus \left(\bigoplus_{E \in \mathcal{E}^H} V_E^{h,m_E} \right) \oplus \left(\bigoplus_{T \in \mathcal{T}^H} V_T^{h,m_T} \right) \tag{12}$$

where $\boldsymbol{m} = (\cup_{E \in \mathcal{E}^H} m_E) \cup (\cup_{T \in \mathcal{T}^H} m_T)$ is the multiindex containing the indices m_E and m_T for alla edges and elements.

The multiscale method is then simply obtained by using this reduced order space in the standard variational formulation: find $v \in V^{h,\boldsymbol{m}}$ such that

$$A(\boldsymbol{U}^{\boldsymbol{m}}, \boldsymbol{v}) = b(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V^{h,\boldsymbol{m}}, \tag{13}$$

4 A POSTERIORI ERROR ESTIMATE

Let $|||\cdot|||$ denote the energy norm, $|||v|||^2 = A(v,v)$ and let $\boldsymbol{U}^h$ denote the standard finite element solution in V^h. Then we have the following a posteriori error estimate

$$|||\boldsymbol{U}^h - \boldsymbol{U}^{\boldsymbol{m}}||| \leq \left(\sum_{E\in\mathcal{E}} \frac{\|\boldsymbol{R}_E(\boldsymbol{U})\|^2}{\Lambda_{E,m_E+1}} + \sum_{T\in\mathcal{T}} \frac{\|\boldsymbol{R}_T(\boldsymbol{U})\|^2}{\Lambda_{T,m_T+1}} \right)^{1/2}. \tag{14}$$

Here the subspace residual $\boldsymbol{R}_I(\boldsymbol{w}) \in V_I^h$, $I \in \mathcal{E}^H \cup \mathcal{T}^H$, is defined by

$$(\boldsymbol{R}_I(\boldsymbol{w}), \boldsymbol{v}) = b(\boldsymbol{v}) - A(\boldsymbol{w}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_I^h, \quad I \in \mathcal{E}^H \cup \mathcal{T}^H \tag{15}$$

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