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HIGH-ORDER MESH GENERATION ON CAD GEOMETRIES

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Abstract. We present a technique to extend Jacobian-based distortion (quality) measures for planar triangles to high-order isoparametric elements of any interpolation degree on CAD parameterized surfaces. The resulting distortion (quality) measures are expressed in terms of the parametric coordinates of the nodes. These extended distortion (quality) measures can be used to check the quality and validity of a high-order surface mesh. We also apply them to simultaneously smooth and untangle high-order surface meshes by minimizing the extended distortion measure. The minimization is performed in terms of the parametric coordinates of the nodes. Thus, the nodes always lie on the surface. Finally, we include several examples to illustrate the application of the proposed technique.

1 Introduction

It is well known that computational methods for solving partial differential equations require domain discretizations composed by valid and high-quality elements [1, 2, 3]. If the mesh contains inverted elements, it can not be used for computational purposes. Moreover, if the mesh does not have a minimum quality, the accuracy of the finite element computation is degraded.

In order to improve the quality of a mesh the nodes can be relocated (smoothing) [4, 5, 6]. Note that in 3D applications, it is of the major importance to ensure a high-quality surface mesh. If a boundary mesh face is inverted, the corresponding mesh element is inverted and cannot be recovered once the surface mesh is fixed. Therefore, in this work we present a technique to extend any Jacobian-based distortion (quality) for planar elements to high-order elements with the nodes on CAD surfaces. The resulting measures are expressed in terms of the parametric coordinates of the surface. We use these measures in order to develop a simultaneous smoothing and untangling technique for high-order meshes with the nodes on a parameterized surface. The resulting meshes are composed by valid and high-quality elements with the nodes on the surface. It is important to highlight that we can ensure that the optimized nodes lie on the original CAD surface and not on an approximation, since the optimization process is written in terms of the parametric coordinates of the mesh nodes.

The proposed technique relies on the framework of algebraic quality measures introduced in [2]. In order to improve the quality of a valid mesh, an optimization approach based on Jacobian-based measures is proposed in [6]. These optimization approaches can also be used to untangle inverted elements. On the one hand, references [7, 8] propose a two-step procedures that first untangle the elements and second smooth the node location. On the other hand, in Reference [9] a simultaneous smoothing and untangling technique for triangular planar meshes is proposed by means of a modification of a Jacobian-based distortion measure. It is worth to notice that this technique has been extended to quadrilateral and hexahedral meshes [10] and to non-planar triangular meshes [11]. The simulatneous smoothing-untangling is the approach selected in this work.

Several techniques have been developed to optimize meshes on surfaces, generally defined by discrete representations, see [11, 12, 13, 14, 15]. However, in our work we consider parameterized CAD geometries and our objective is to ensure that during the optimization process the nodes are always located on the surface. In [16] we already proposed to quantify the distortion (quality) of a linear surface element in terms of the coordinates on the parametric space of the CAD surface. An optimization approach based on the proposed distortion measure ensures that the nodes always lie on the surface, since the whole process is developed in the parametric space of the original surface.

Several methods have been proposed to generate high-order planar or 3D meshes, see [17, 18, 19, 20, 21]. The standard approach to generate a high-order mesh consists on an a-posteriori procedure composed by three steps: (1) generate a linear mesh; (2) increase the order of the elements and curve them to fit the boundary; and (3) optimize the node locations so that the mesh is valid and is composed by high-quality elements. The method proposed in this paper relies on the work developed for planar high-order elements presented in [18]. Specifically, we propose to extend the measures for planar high-order elements presented in [18] to high-order meshes on parametrized surfaces using the framework presented in [16], where planar measures for linear elements are extended to surfaces.

The outline of the paper is as follows. First, in Section 2, we review the definition of distortion measure for planar elements presented in [18]. Next in Section 3, we present the formulation to extend any Jacobian distortion measure for linear triangles to high-order

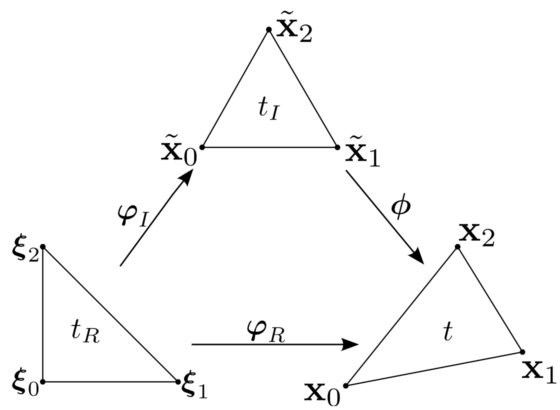

Figure 1: Mappings between the reference, the ideal and the physical elements.

elements with the nodes on parameterized surfaces. Afterwards, in Section 4, we detail the optimization procedure in terms of the parametric coordinates. We develop a non-linear least-squares problem in order to enforce the ideal configuration for the elements of the surface mesh. Finally, we present several examples to show the applications of the proposed method, Section 5.

2 Preliminaries

In this section, we first review the family of Jacobian-based distortion measures, presented in [2]. Second, we summarize the definition of distortion measure for planar high-order elements presented in [18], in which it is shown how to extend the Jacobian-based measures for linear triangles to planar high-order elements.

Let η be a Jacobian-based distortion measure for planar elements [2], with image $[1,\infty)$, taking value 1 for an ideal configuration of the element, and value ∞ when it is degenerated or tangled. Let q be the corresponding quality measure, defined as

$$q = \frac{1}{\eta}. \tag{1}$$

The image of the quality measure q is $[0,1]$, taking value 1 for ideal configurations and 0 for degenerated ones. Our goal is to extend these measures to qualify high-order elements on parameterized surfaces.

2.1 Jacobian-based distortion measures for planar linear triangle elements

In order to determine the quality of a high-order element t on a parameterized surface, we generalize the Jacobian based quality measures for linear elements [2]. To this end, we consider a mapping $\boldsymbol{\phi}$ from the ideal element t_I to the physical element t, see Figure 1. To determine this mapping, we consider the isoparametric mappings $\boldsymbol{\varphi}_R$ (from the reference element t_R to t) and $\boldsymbol{\varphi}_I$ (from t_R to t_I). For linear triangles, these mappings are affine.

Name	Distortion measure $\eta(\mathbf{S})$
Shape measure	$\eta(\mathbf{S}) = \dfrac{\|\|\mathbf{S}\|\|^2}{d \cdot \sigma(\mathbf{S})^{2/d}}$
Oddy et al. measure	$\eta(\mathbf{S}) = \dfrac{3}{d}\sigma^{-4/d}(\mathbf{S})\left(\|\|\mathbf{S}^T\mathbf{S}\|\|^2 - \dfrac{1}{3}\|\|\mathbf{S}\|\|^4\right)$

Table 1: Algebraic distortion measures for linear elements

The mapping between the ideal and the physical element is determined by

$$\boldsymbol{\phi} = \boldsymbol{\varphi}_R \circ \boldsymbol{\varphi}_I^{-1}. \tag{2}$$

Note that $\boldsymbol{\phi}$ is also an affine mapping, since $\boldsymbol{\varphi}_I^{-1}$ and $\boldsymbol{\varphi}_R$ are so. For linear elements it is usual to define a distortion measure in terms of the Jacobian matrix $\mathbf{S} := \mathbf{D}\boldsymbol{\phi}$. These distortion measures, herein denoted by $\eta(\mathbf{S})$, quantify a specific type of distortion of the physical element in a range scale $[1, \infty)$. Several distortion measures for linear triangles have been proposed in literature, see [2]. In Table 1 we present two distortion measures that we use to test the proposed high-order quality. Parameter d is the number of spatial dimensions, $\sigma(\mathbf{S})$ is the determinant of $\mathbf{S}$, and $||\mathbf{S}|| = \sqrt{\mathrm{tr}(\mathbf{S}^t\mathbf{S})}$ is its Frobenius norm.

2.2 Distortion measure for planar high-order elements

Let t be a nodal high-order element of order p determined by n_p nodes with coordinates $\mathbf{x}_i \in \mathbb{R}^{d_x}$, for $i = 1, \ldots, n_p$ and being d_x the physical space dimension. Given a reference element t_R with nodes $\boldsymbol{\xi}_j \in \mathbb{R}^{d_\xi}$, being $j = 1, \ldots, n_p$ and d_ξ the reference space dimension, we consider the basis $\{N_i\}_{i=1,\ldots,n_p}$ of nodal shape functions (Lagrange interpolation) of order p. In this basis, the high-order isoparametric mapping from t_R to t can be expressed as:

$$\begin{array}{rccl} \boldsymbol{\varphi}_R: & t_R \subset \mathbb{R}^d & \longrightarrow & t \subset \mathbb{R}^d \\ & \boldsymbol{\xi} & \longmapsto & \mathbf{x} = \boldsymbol{\varphi}_R(\boldsymbol{\xi}; \mathbf{x}_1, \ldots, \mathbf{x}_{n_p}) = \displaystyle\sum_{i=1}^{n_p} \mathbf{x}_i N_i(\boldsymbol{\xi}), \end{array} \tag{3}$$

where $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^{d_\xi})^T$ and $\mathbf{x} = (x^1, \ldots, x^{d_x})^T$. Note that the shape functions $\{N_i\}_{i=1,\ldots,n_p}$ depend on the selection of $\boldsymbol{\xi}_j$, for $j = 1, \ldots, n_p$. In addition, they form a partition of the unity on t_R, and hold that $N_i(\boldsymbol{\xi}_j) = \delta_{ij}$, for $i, j = 1, \ldots, n_p$. In this paper we focus on nodal high-order triangular elements of order p, but the same approach is valid for quadrilaterals. Hence, the number of nodes n_p is $\frac{1}{2}(p+1)(p+2)$, and the space dimensions for planar meshes are $d_\xi = d_x = 2$. Therefore, the Jacobian of the isoparametric mapping (3) is a $d_x \times d_\xi = 2 \times 2$ matrix.

To define the high-order distortion measure of the physical element, we have to select first the ideal element t_I and a distribution of points. Herein, we choose a straight-sided equilateral triangle as the ideal element. In addition, we select the desired distribution of the nodes on the ideal element (*e.g.* equi-distributed or Fekete points). In general the mappings $\boldsymbol{\varphi}_I$ and $\boldsymbol{\varphi}_R$, see Equation (3), are not affine. Hence, $\boldsymbol{\phi} = \boldsymbol{\varphi}_R \circ \boldsymbol{\varphi}_I^{-1}$ is also not affine, and the Jacobian matrix is not constant. The expression of the Jacobian is:

$$\mathbf{D}\boldsymbol{\phi}(\tilde{\mathbf{x}}; \mathbf{x}_1, \ldots, \mathbf{x}_{n_p}) = \mathbf{D}\boldsymbol{\varphi}_R(\boldsymbol{\varphi}_I^{-1}(\tilde{\mathbf{x}}); \mathbf{x}_1, \ldots, \mathbf{x}_{n_p}) \cdot \mathbf{D}\boldsymbol{\varphi}_I^{-1}(\tilde{\mathbf{x}}) \tag{4}$$

where $\tilde{\mathbf{x}}$ is a point on the ideal element.

Similar to the linear element case, we define a distortion measure based on the Jacobian matrix of $\boldsymbol{\phi}$. However, the Jacobian of the elements is not constant. Nevertheless, the Jacobian on a point allows measuring the local deviation between the ideal and the physical element. Thus, we can obtain an elemental distortion measure by integrating the Jacobian based distortion measure on the whole ideal element.

Definition 1 *The* high-order distortion measure for a high-order planar element *with nodes* $\mathbf{x}_1, \ldots, \mathbf{x}_{n_p}$ *is*

$$\eta_{\boldsymbol{\phi}}(\mathbf{x}_1, \ldots, \mathbf{x}_{n_p}) := \left(\frac{1}{|t_I|} \int_{t_I} \eta^2(\mathbf{D}\boldsymbol{\phi}(\tilde{\mathbf{x}}; \mathbf{x}_1, \ldots, \mathbf{x}_{n_p}))\ d\mathbf{x} \right)^{\frac{1}{2}}, \tag{5}$$

where η is a distortion measure for linear elements based on the Jacobian matrix of the representation of the element, and $|t_I|$ is the area of the element element.

The *high-order quality measure for a high-order planar element* is $q_{\boldsymbol{\phi}} := 1/\eta_{\boldsymbol{\phi}}$, see [18] for an extended analysis for planar elements.

3 Distortion measure for high-order elements on parameterized surfaces

In this section, we first develop an analytical formulation to extend any Jacobian-based distortion measure for planar triangles $\eta_{\boldsymbol{\phi}}$, see Equation (5), to high-order elements with nodes on a parameterized surface Σ. As a result, we obtain a quality measure expressed in terms of the coordinates of the nodes in the parametric space of the surface.

3.1 Definitions

Assume that the surface Σ is parameterized by a continuously differentiable and invertible mapping

$$\begin{array}{rccc} \boldsymbol{\varphi}: & \mathcal{U} \subset \mathbb{R}^2 & \longrightarrow & \Sigma \subset \mathbb{R}^3 \\ & \mathbf{u} = (u, v) & \longmapsto & \mathbf{x} = \boldsymbol{\varphi}(\mathbf{u}), \end{array} \tag{6}$$

where $\mathcal{U}$ is the parametric space of the surface. In this work, we use OpenCASCADE library [22] to retrieve the parameterization of the surfaces from the CAD model.

Similarly to the planar case, Equation (5), our objective is to quantify the distortion of the tangent vectors in each point of the surface elements. However, the tangent vectors on a point of the surface element live in the tangent plane, that is immersed in $\mathbb{R}^3$. Specifically, the Jacobian of the isoparametric mapping is a $d_x \times d_\xi = 3 \times 2$ not square

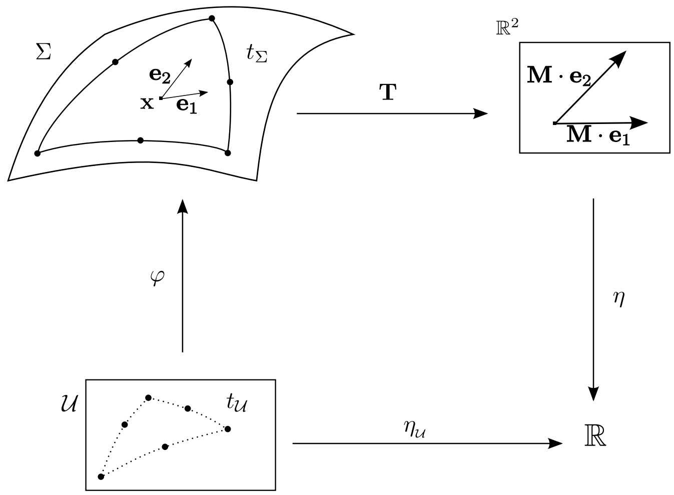

Figure 2: Diagram of mappings involved in the definition of the quality measure.

matrix. Therefore, we propose to define an embedding $\mathbf{T}$ from the tangent space on a point $\mathbf{x} = \boldsymbol{\phi}(\tilde{\mathbf{x}})$ of the surface element to $\mathbb{R}^2$, see Figure 2. Specifically, we define $\mathbf{T}$ as:

$$\begin{array}{rccc} \mathbf{T}: & \mathbb{R}^3 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times \mathbb{R}^2 \\ & \mathbf{D}\boldsymbol{\phi}(\tilde{\mathbf{x}}) & \longrightarrow & \mathbf{M} \cdot \mathbf{D}\boldsymbol{\phi}(\tilde{\mathbf{x}}), \end{array} \tag{7}$$

where $\mathbf{M}$ is a matrix composed by the two vectors corresponding to the basis derived from the Gram-Schmidt process applied to vectors $\mathbf{e}_1 := \frac{\partial \phi}{\partial \tilde{x}^1}$ and $\mathbf{e}_2 := \frac{\partial \phi}{\partial \tilde{x}^2}$. Hence, $\mathbf{M} = [\tilde{\mathbf{e}}_1 \quad \gamma \tilde{\mathbf{e}}_2]^T$, where $\tilde{\mathbf{e}}_i,\ i = 1,2$ are the Gram-Schmidt orthonormal vectors, and $\gamma = \pm 1$ is determined to ensure a well oriented basis. Note that $\mathbf{T}(\mathbf{D}\boldsymbol{\phi})$ is a 2×2 matrix to which we can apply the Jacobian-based distortion measures presented in Section 2.1.

Finally, using the embedding (7) we can express the distortion and quality measures of the surface elements in terms of the parametric coordinates of the nodes, see Figure 2:

Definition 2 *The* distortion measure for a high-order element on parametric coordinates *with nodes* $\mathbf{u}_1, \ldots, \mathbf{u}_{n_p} \in \mathcal{U}$ *is*

$$\eta_\mathcal{U}(\mathbf{u}_1, \ldots, \mathbf{u}_{n_p}) := \left(\frac{1}{|t_I|} \int_{t_I} \eta^2 \left(\mathbf{T} \left(\mathbf{D}\boldsymbol{\phi} \left(\tilde{\mathbf{x}}; \boldsymbol{\varphi}(\mathbf{u}_1), \ldots, \boldsymbol{\varphi}(\mathbf{u}_{n_p}) \right) \right) \right) d\mathbf{x} \right)^{\frac{1}{2}}. \tag{8}$$

Analogously, the quality measure for a high-order element on parametric coordinates is $q_\mathcal{U} := 1/\eta_\mathcal{U}$.

4 Application to high-order mesh optimization

In this section, we present an algorithm to optimize the distortion (quality) measure of triangular high-order meshes. It is important to point out that we want to ensure that the nodes lie always on the surface. Therefore, the optimization approach is developed in the parametric space and the result is mapped to the surface by means of the parameterization.

The main goal of a simultaneous smoothing and untangling method is to obtain high-quality meshes composed by valid (non-inverted) elements. Note that the best possible result, can be characterized in terms of the distortion measure. That is, given a distortion measure η and a mesh $\mathcal{M}$ composed by n_N nodes and n_E elements, the node location is ideal if

$$\eta(\mathbf{T}(\mathbf{D}\boldsymbol{\phi}_j(\tilde{\mathbf{x}};\boldsymbol{\varphi}(\mathbf{u}_{j_1}),\ldots,\boldsymbol{\varphi}(\mathbf{u}_{j_{n_p}})))) = 1, \qquad \forall \tilde{\mathbf{x}} \in t_{I_j},\ j = 1,\ldots,n_E, \tag{9}$$

where $e_j = (\boldsymbol{\varphi}(\mathbf{u}_{j_1}),\ldots,\boldsymbol{\varphi}(\mathbf{u}_{j_{n_p}}))$ is the jth element, t_{I_j} is the ideal element corresponding to e_j, and $\boldsymbol{\phi}_j$ is the mapping between the jth ideal and physical elements. However, for a fixed mesh topology and a given surface the node location that leads to an ideal mesh distortion is not in general achievable. That is, the constrains in Equation (9) cannot be imposed strongly and therefore, we just enforce the ideal mesh distortion in the least-squares sense.

For a given mesh topology and a set of fixed nodes (nodes on the boundary of the domain), we formulate the least-squares problem in terms of the coordinates of a set of free nodes (nodes in the interior of the domain). To this end, and without loss of generality, we reorder the coordinates of the nodes, $\mathbf{u}_i$, in such a way that $i = 1,\ldots,n_F$ are the indices corresponding to the free nodes, and $i = n_F + 1,\ldots,n_N$ correspond to the fixed nodes. Thus, we can formulate the mesh optimization problem as

$$\min_{\mathbf{u}_1,\ldots,\mathbf{u}_{n_F}} f(\mathbf{u}_1,\ldots,\mathbf{u}_{n_F};\mathbf{u}_{n_F+1},\ldots,\mathbf{u}_{n_N}), \tag{10}$$

where f is the objective function, defined as:

$$f(\mathbf{u}_1,\ldots,\mathbf{u}_{n_F};\mathbf{u}_{n_F+1},\ldots,\mathbf{u}_{n_N}) := \sum_{j=1}^{n_E}\int_{t_{I_j}} (\eta(\mathbf{T}(\mathbf{D}\boldsymbol{\phi}_j(\tilde{\mathbf{x}};\boldsymbol{\varphi}(\mathbf{u}_{j_1}),\ldots,\boldsymbol{\varphi}(\mathbf{u}_{j_{n_p}})))) - 1)^2 \mathrm{d}\tilde{\mathbf{x}}.$$

In this work we illustrate the distortion and quality measures for high-order elements using the shape distortion measure presented Table 1. In order to untangle invalid meshes in a continuous optimization procedure, we use the modification of the Jacobian-based distortion measure presented in [9, 18].

5 Examples: mesh generation on CAD geometries

In this section we illustrate the overall process to generate a high-order mesh on a CAD geometry. Specifically, we select two different CAD geometries: a Falcon aircraft, Figure 3, and a component of a gear box, Figure 4. For each example geometry we show the complete sequence of steps of the a posteriori procedure to generate a valid high-order mesh:

Fig.	Min.Q.	Max.Q.	Mean Q.	Std.Dev.	Tang.
3(a)	0.21	1.00	0.93	0.09	0
3(c)	0.00	1.00	0.92	0.10	5
3(e)	0.25	1.00	0.97	0.06	0
4(a)	0.53	1.00	0.84	0.16	0
4(b)	0.00	1.00	0.73	0.34	130
4(c)	0.52	1.00	0.84	0.16	0

Table 2: Shape quality statistics of the meshes presented in Figure 3.

1. **A linear mesh is generated on the geometry**: Figure 3(a) shows the initial mesh generated on the Falcon aircraft, and Figure 4(a) the mesh generated on the gear box.

2. **The order of the mesh elements is increased**: We define a high-order node distribution for each element on the parametric space, and we map it to the surface. Note that for each geometry we have selected a different order. For instance, we use elements of order 3 for the Falcon aircraft, and elements of order 10 for the component of the gear box. In this step, tangled elements can be generated due to two main reasons:

    - The boundary elements can have auto-intersections due to the fact that in the parametric space the boundary edges are curved to fit the geometry, but the inner edges are maintained straight. This phenomena can be observed in Figure 4(b), .
    - If the quality of the parameterization is low, the composition of the high-order distribution on the parametric space together with the parameterization can lead to an invalid node distribution on the parametric space. This issue appears in the nose of the aircraft in Figure 3(d).

3. **The high-order mesh is optimized**: We apply the smoothing-untangling approach presented in Section 4 to the meshes. Figures 3(e) and 4(c) show the resulting meshes for each geometry.

The distortion measure selected in the presented examples is the shape distortion measure, detailed in Table 1. Table 2 details the quality statistics for each one of the presented meshes. Note that the obtained meshes are composed of high-quality elements. In all the cases we have untangled the initial inverted elements, and achieved a final high-quality configuration.

6 Concluding remarks

In this paper, we first detail a new technique to extend any distortion (quality) measure defined for planar elements to parameterized surfaces. Next, we develop an optimization

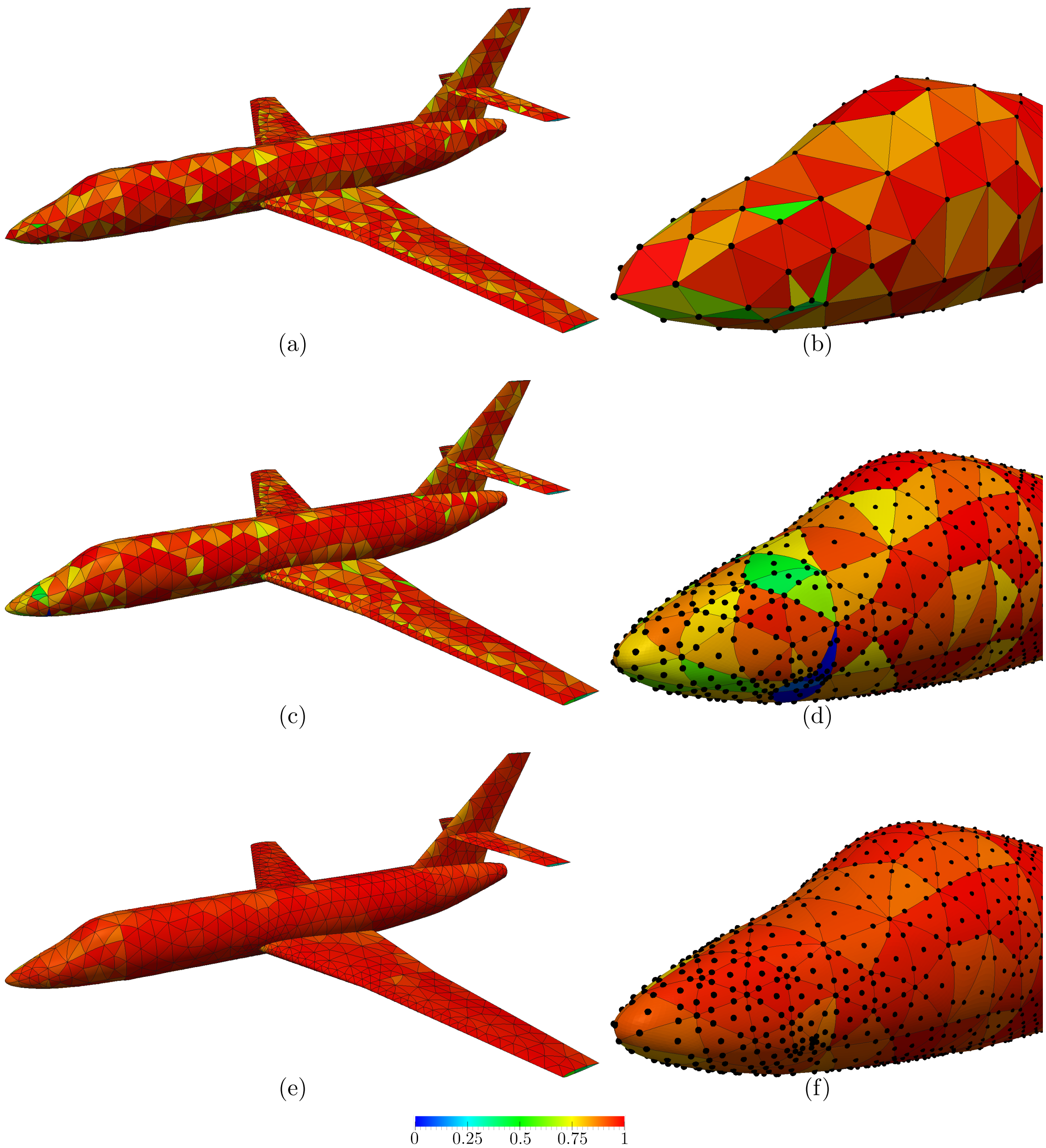

Figure 3: Order 3 mesh for a Falcon aircraft. The elements are colored according to the shape quality measure. (a,b) Initial linear mesh. (c,d) Initial order 3 mesh obtained after increase the order of the initial linear mesh. (e,f) Optimized order 3 mesh.

procedure to smooth and untangle meshes on parameterized surfaces. It is important to point out that the proposed measure expresses the quality of the elements on the surface in terms of the parametric coordinates of its nodes. Therefore, the optimization procedure

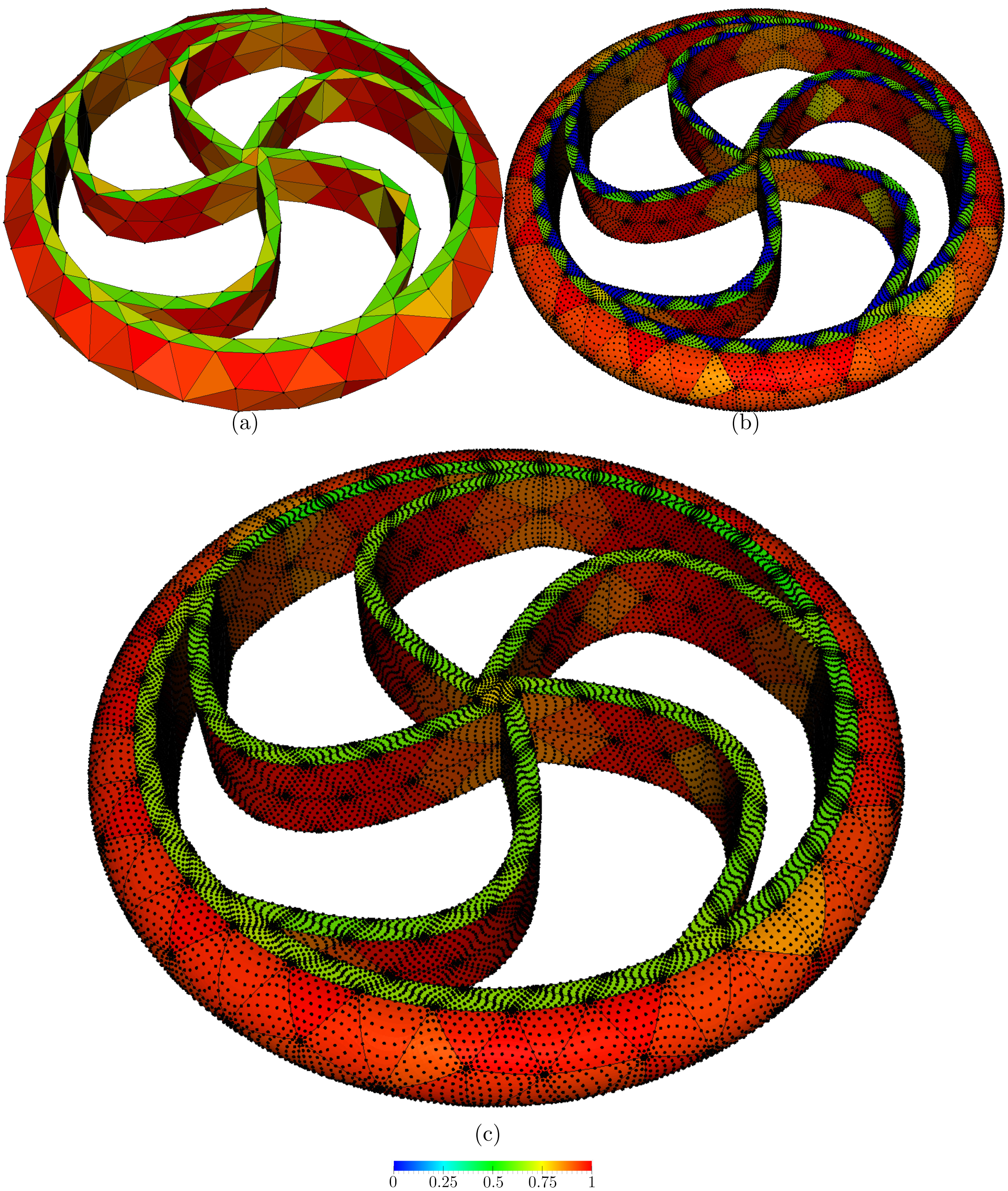

Figure 4: Order 10 mesh for a component of a gear box. The elements are colored according to the shape quality measure. (a,b) Initial linear mesh. (c,d) Initial order 10 mesh obtained after increase the order of the initial linear mesh. (e,f) Optimized order 10 mesh.

is also written in terms of the parametric coordinates. Hence, it ensures that the nodes are always placed on the surface. Finally, in the presented examples we have illustrated the mesh generation procedure with two different CAD geometries and with two different orders.

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