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ERROR ASSESSMENT FOR TIMELINE-DEPENDENT QUANTITIES OF INTEREST IN TRANSIENT ELASTODYNAMICS

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Abstract. This work presents a new approach to assess the error in specific quantities of interest in the framework of linear elastodynamics. In particular, a new type of quantities of interest (referred as timeline-dependent quantities) is proposed. These quantities are scalar time-dependent outputs of the transient solution which are better suited to time-dependent problems than the standard scalar ones available in the literature. The proposed methodology furnishes error estimates for both the standard scalar and the new timeline-dependent quantities of interest. The key ingredient is the modal-based approximation of the associated adjoint problems which allows efficiently computing and storing the adjoint solution.

1 INTRODUCTION

Assessing the reliability and/or improving the efficiency of the finite element based approximations has motivated the development of a huge variety of error assessment techniques. The pioneering references on this topic focus in steady-state elliptic problems, e.g. linear elasticity or steady heat transfer. In the context of elliptic problems, the early works consider the energy norm as an error measure [1, 2, 3]. Much later, functionals outputs or *quantities of interest* are introduced to assess the error [4, 5, 6, 7]. The estimates assessing the error in quantities of interest are usually referred in the literature as *goal-oriented*. These techniques are extended to deal with other linear and non-linear problems, as well as to time-dependent problems.

An important issue associated with goal-oriented estimates for elastodynamics (and also for other time-dependent problems) is the definition of the quantity of interest itself. Typically, the quantity is expressed in terms of a (linear) functional, which transforms the solution of the problem into a single representative scalar value. In many cases, a single scalar value does not provide enough pieces of information about the whole time-space solution. This suggests introducing a new type of quantities of interest. The output of such a quantity of interest is not anymore a scalar quantity but a time-dependent function. The major novelty of this article is the introduction of this new type of quantities. They are referred as *timeline-dependent* quantities of interest in contrast with the standard *scalar* quantities.

2 PROBLEM STATEMENT

2.1 Governing equations

Consider a visco-elastic body occupying an open bounded domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, with boundary $\partial\Omega$. The boundary is divided in two disjoint parts, Γ_{N} and Γ_{D} such that $\partial\Omega = \overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{D}}$ and the time interval under consideration is $I := [0, T]$. Under the assumption of small perturbations, the evolution of displacements $\mathbf{u}(\mathbf{x}, t)$ and stresses $\boldsymbol{\sigma}(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$ and $t \in I$, is described by the visco-elastodynamic equations,

$$
\begin{aligned}
\rho(\ddot{\mathbf{u}} + a_1\dot{\mathbf{u}}) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} &= \mathbf{f} \quad \text{in } \Omega \times I, && \text{(1a)}\\
\mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}} \times I, && \text{(1b)}\\
\boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{g} \quad \text{on } \Gamma_{\mathrm{N}} \times I, && \text{(1c)}\\
\mathbf{u} &= \mathbf{u}_0 \quad \text{at } \Omega \times \{0\}, && \text{(1d)}\\
\dot{\mathbf{u}} &= \mathbf{v}_0 \quad \text{at } \Omega \times \{0\}. && \text{(1e)}
\end{aligned}
$$

where an upper dot indicates partial derivation with respect to time, that is $\dot{(\bullet)} := \frac{\mathrm{d}}{\mathrm{d}t}(\bullet)$, and $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. The problem data are the mass density $\rho = \rho(\mathbf{x}) > 0$, the first Rayleigh coefficient $a_1 \geq 0$, the body force $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ and the traction $\mathbf{g} = \mathbf{g}(\mathbf{x}, t)$ acting on the Neumann boundary $\Gamma_{\mathrm{N}} \times I$. The initial conditions for displacements and velocities are $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x})$ and $\mathbf{v}_0 = \mathbf{v}_0(\mathbf{x})$ respectively. For the sake of simplicity and without any loss of generality, Dirichlet conditions (1b) are taken as homogeneous.

The set of equations (1) is closed with the constitutive law,

$$
\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}(\mathbf{u} + a_2\dot{\mathbf{u}}), \tag{2}
$$

where the parameter $a_2 \geq 0$ is the second Rayleigh coefficient, the tensor $\boldsymbol{\mathcal{C}}$ is the standard 4th-order elastic Hooke tensor and the kinematic relation (corresponding to small perturbations) $\boldsymbol{\varepsilon}(\mathbf{w}) := \frac{1}{2}(\boldsymbol{\nabla}\mathbf{w} + \boldsymbol{\nabla}^{\mathrm{T}}\mathbf{w})$ is considered.

2.2 Numerical approximation

In the following developments, $\hat{\mathbf{u}}$ is assumed to be an approximation of the solution of the boundary value problem (1). For technical reasons, $\hat{\mathbf{u}}$ must have C^0-continuity in space and C^1-continuity in time. Most typically, the approximation computed with the standard Newmark method, say $\mathbf{u}^{H,\Delta t}$, does not fulfill these continuity requirements and has to be post-processed to obtain a suitable smooth in time function $\hat{\mathbf{u}}$.

The numerical approximation $\hat{\mathbf{u}}$ is computed here as a post process of the Newmark solution using the method of the linear accelerations [8]. This post-process consist basically in integrate in time a piecewise linear interpolation of the Newmark accelerations furnishing the smooth velocity $\dot{\hat{\mathbf{u}}}$ and then integrating in time again furnishing the smooth displacement $\hat{\mathbf{u}}$, see [8, 9] for details.

2.3 Scalar and timeline-dependent quantities of interest

A quantity of interest is represented by a functional $L^{\mathcal{O}}(\cdot)$ extracting a single scalar value, $s_T := L^{\mathcal{O}}(\mathbf{u}) \in \mathbb{R}$, of the space-time solution $\mathbf{u}$. A typical expression for this functional is given by

$$L^{\mathcal{O}}(\mathbf{u}) := \int_0^T (\mathbf{f}^{\mathcal{O}}(t), \dot{\mathbf{u}}(t))\ \mathrm{d}t + \int_0^T (\mathbf{g}^{\mathcal{O}}(t), \dot{\mathbf{u}}(t))_{\Gamma_\mathrm{N}}\ \mathrm{d}t + (\rho \mathbf{v}^{\mathcal{O}}, \dot{\mathbf{u}}(T)) + a(\mathbf{u}^{\mathcal{O}}, \mathbf{u}(T)), \quad (3)$$

where $\mathbf{f}^{\mathcal{O}}$, $\mathbf{g}^{\mathcal{O}}$, $\mathbf{v}^{\mathcal{O}}$ and $\mathbf{u}^{\mathcal{O}}$ are the data characterizing the quantity of interest. The functions $\mathbf{f}^{\mathcal{O}}$ and $\mathbf{g}^{\mathcal{O}}$ extract global or localized averages of velocities in Ω and Γ_N, respectively, over the whole time simulation $[0, T]$ whereas $\mathbf{v}^{\mathcal{O}}$ and $\mathbf{u}^{\mathcal{O}}$ assess averages of velocities and strains or displacements respectively at the final simulation time T.

The quantity of interest associated with the adjoint solution, namely s_T, is obviously unknown and it is approximated by the quantity of interest associated with the approximated solution $\hat{\mathbf{u}}$, that is $s_T \approx \hat{s}_T := L^{\mathcal{O}}(\hat{\mathbf{u}})$. Goal oriented error estimates aims at assessing the quality of the approximation $\hat{s}_T$ by means of approximating the error $s_T^\mathrm{e} = s_T - \hat{s}_T$. Consequently, the problem of goal-oriented consists in finding approximations of the value s^e.

This work extends the paradigm of classical goal-oriented error estimation by introducing the new concept of *timeline-dependent quantities of interest*. Timeline-dependent quantities of interest are defined as an extension of (3) as

$$L_{\mathrm{TL}}^{\mathcal{O}}(\mathbf{u})(t) := \int_0^t (\mathbf{f}^{\mathcal{O}}(\tau), \dot{\mathbf{u}}(\tau))\ \mathrm{d}\tau + \int_0^t (\mathbf{g}^{\mathcal{O}}(\tau), \dot{\mathbf{u}}(\tau))_{\Gamma_\mathrm{N}}\ \mathrm{d}\tau + (\rho \mathbf{v}^{\mathcal{O}}, \dot{\mathbf{u}}(t)) + a(\mathbf{u}^{\mathcal{O}}, \mathbf{u}(t)). \quad (4)$$

Note that the time-line dependent quantity $s(t) := L_{\mathrm{TL}}^{\mathcal{O}}(\mathbf{u})(t)$ associated with the function $\mathbf{u}$ is a time dependent function instead of a single scalar value, see figure 1.

The aim of timeline-dependent goal-oriented error estimation strategies is assessing the quality of $\hat{s}(t) = L_{\mathrm{TL}}^{\mathcal{O}}(\hat{\mathbf{u}}; t)$, that is the difference between the exact quantity of interest $s(t) = L_{\mathrm{TL}}^{\mathcal{O}}(\mathbf{u}; t)$ and the approximation obtained with the numerical simulation $\hat{s}(t)$.

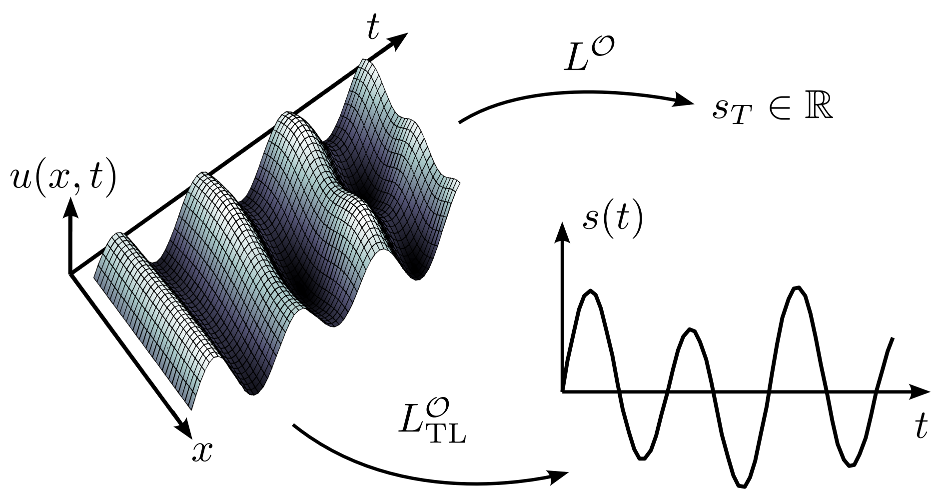


Figure 1: Illustration of scalar and timeline-dependent quantities of interest. The functional $L^{\mathcal{O}}$ maps the time-space solution $\mathbf{u}$ into a scalar value $s_T \in \mathbb{R}$. The operator $L^{\mathcal{O}}_{\mathrm{TL}}$ transforms $\mathbf{u}$ into a time-dependent function $s(t)$.

Thus, the goal of goal-oriented error estimates for timeline-dependent quantities is finding approximation of the time-dependent function

$$s^{\mathrm{e}}(t) := s(t) - \hat{s}(t).$$

3 ASSESSING SCALAR AND TIMELINE-DEPENDENT QUANTITIES OF INTEREST

This section is devoted to present a novel approach to assess the error both in the scalar quantity of interest, s^{e}_T, and in the timeline-dependent quantity, $s^{\mathrm{e}}(t)$, using the modal analysis to obtain a proper approximation of the adjoint solution.

3.1 Assessing Scalar quantities

Assessing the error in quantities of interest requires introducing an auxiliary problem associated with the functional $L^{\mathcal{O}}(\cdot)$, usually denoted by *adjoint* or *dual* problem [9]. The strong form of the adjoint problem associated with the quiantity defined in (3) is, see [9] for details,

$$\rho(\ddot{\mathbf{u}}^{\mathrm{d}} - a_1\dot{\mathbf{u}}^{\mathrm{d}}) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathrm{d}} = -\mathbf{f}^{\mathcal{O}} \quad \text{in } \Omega\times I, \tag{5a}$$

$$\mathbf{u}^{\mathrm{d}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}\times I, \tag{5b}$$

$$\boldsymbol{\sigma}^{\mathrm{d}}\cdot\mathbf{n} = -\mathbf{g}^{\mathcal{O}} \quad \text{on } \Gamma_{\mathrm{N}}\times I, \tag{5c}$$

$$\mathbf{u}^{\mathrm{d}} = \mathbf{u}^{\mathcal{O}} \quad \text{at } \Omega\times\{T\}, \tag{5d}$$

$$\dot{\mathbf{u}}^{\mathrm{d}} = \mathbf{v}^{\mathcal{O}} \quad \text{at } \Omega\times\{T\}, \tag{5e}$$

with the constitutive law

$$\boldsymbol{\sigma}^{\mathrm{d}} := \boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{d}} - a_2\dot{\mathbf{u}}^{\mathrm{d}}). \tag{6}$$

Note that the terms affected by a_1 and a_2 have opposite sign that the ones in the original problem (1). Consequently, the adjoint problem has to be integrated backwards in time, starting from the *final conditions* (5d) and (5e).

The solution of the adjoint problem $\mathbf{u}^{\mathrm{d}}$ allows representing the error in the quantity of interest in terms of residuals. That is

$$L^{\mathcal{O}}(\hat{\mathbf{e}}) = R(\mathbf{u}^{\mathrm{d}}) \tag{7}$$

where $R(\cdot) := L(\cdot) - B(\hat{\mathbf{u}}, \cdot)$ is the weak residual associated with the numerical approximation $\hat{\mathbf{u}}$. The forms $B(\cdot,\cdot)$ and $L(\cdot)$ are defiend as

$$B(\mathbf{v},\mathbf{w}) := \int_I (\rho(\ddot{\mathbf{v}}+a_1\dot{\mathbf{v}}), \dot{\mathbf{w}})\ \mathrm{d}t + \int_I a(\mathbf{v}+a_2\dot{\mathbf{v}}, \dot{\mathbf{w}})\ \mathrm{d}t + (\rho\dot{\mathbf{v}}(0^+), \dot{\mathbf{w}}(0^+)) + a(\mathbf{v}(0^+), \mathbf{w}(0^+)),$$

and

$$L(\mathbf{w}) := \int_I l(t;\dot{\mathbf{w}}(t))\ \mathrm{d}t + (\rho\mathbf{v}_0, \dot{\mathbf{w}}(0^+)) + a(\mathbf{u}_0, \mathbf{w}(0^+)).$$

where the standard linear and bilinear forms are introduced

$$a(\mathbf{v},\mathbf{w}) := \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}(\mathbf{w})\ \mathrm{d}\Omega \quad , \quad l(t;\mathbf{w}) := (\mathbf{f}(t), \mathbf{w}) + (\mathbf{g}(t), \mathbf{w})_{\Gamma_{\mathrm{N}}},$$

along with the scalar products

$$(\mathbf{v},\mathbf{w}) := \int_\Omega \mathbf{v}\cdot\mathbf{w}\ \mathrm{d}\Omega \quad \text{and} \quad (\mathbf{v},\mathbf{w})_{\Gamma_{\mathrm{N}}} := \int_{\Gamma_{\mathrm{N}}} \mathbf{v}\cdot\mathbf{w}\ \mathrm{d}\Gamma.$$

Note that the error representation (7) allows obtaining the error in the quantity of interest provided that the exact solution of the adjoint problem is available. Conversely, if an accurate approximation of the adjoint solution is available, say $\tilde{\mathbf{u}}^{\mathrm{d}}$, the error in the quantity of interest is estimated as

$$s^e_T = L^{\mathcal{O}}(\hat{\mathbf{e}}) \approx \hat{R}(\tilde{\mathbf{u}}^{\mathrm{d}}) =: \tilde{s}^e_T. \tag{8}$$

As previously announced, the adjoint problem (5) is of the same type as the original one (1). Thus the adjoint approximation $\tilde{\mathbf{u}}^{\mathrm{d}}$ can be solved with any of the approximation methods available for elastodynamics. Here, the adjoint approximation is computed with modal analysis. For particular quantities of interest, modal analysis is a very efficient way to compute the adjoint problem. Moreover, the modal description of the adjoint solution is a key ingredient in assessing the error in timeline-quantities of interest.

The modal analysis requires computing the M fist vibration modes $\mathbf{q}_i^H$ and frequencies ω_i^H of the problem, $i = 1, \dots, M$, solution of the generalized eigenvalue problem: find $\mathbf{q}_i^H \in \boldsymbol{\mathcal{V}}_0^H$ such that

$$a(\mathbf{q}^H, \mathbf{w}) = (\omega^H)^2(\rho \mathbf{q}^H, \mathbf{w}) \quad \forall \mathbf{w} \in \boldsymbol{\mathcal{V}}_0^H, \tag{9}$$

where $\boldsymbol{\mathcal{V}}_0^H$ is the finite element space (H stands for characteristic element size of the underlying computational mesh). Eigenpairs are sorted from low to high frequencies, namely $\omega_1^H \leq \omega_2^H \cdots \leq \omega_{N_{\mathrm{dof}}}^H$, and eigenvectors are normalized to be orthonormal with respect the product $(\rho\cdot, \cdot)$, i.e.

$$(\rho \mathbf{q}_i^H, \mathbf{q}_j^H) = \delta_{ij}, \quad 1 \leq i, j \leq N_{\mathrm{dof}}. \tag{10}$$

For thechnical reasons (Galerkin cancellation), the adjoint approximation $\tilde{\mathbf{u}}^{\mathrm{d}}$ cannot be computed by means of the eigenpairs $(\mathbf{q}_i^H, \omega_i^H)$. The reason in the eigenvectors have to belong to a richer space than $\boldsymbol{\mathcal{V}}_0^H$. For that reason, new enhanced eigenpairs $(\tilde{\mathbf{q}}_i, \tilde{\omega}_i)$ are computed starting form the original ones $(\mathbf{q}_i^H, \omega_i^H)$ using the post-processing technique proposed in [10]. Once the enhanced eigenpairs are available, the adjoint approximation is computed as the expansion of the enhanced eigenvectors

$$\tilde{\mathbf{u}}^{\mathrm{d}}(\mathbf{x}, t) := \sum_{i=1}^{M} \tilde{\mathbf{q}}_i(\mathbf{x}) \tilde{y}_i(t). \tag{11}$$

Finaly, the time dependent coeficciets are computed solving the scalar ordinary differential equations

$$\ddot{\tilde{y}}_i - [a_1 + a_2(\tilde{\omega}_i)^2]\dot{\tilde{y}}_i + (\tilde{\omega}_i)^2 \tilde{y}_i = \tilde{l}_i, \tag{12a}$$

$$\tilde{y}_i(T) = \tilde{u}_i, \tag{12b}$$

$$\dot{\tilde{y}}_i(T) = \tilde{v}_i, \tag{12c}$$

where $\tilde{l}_i(t) := (\mathbf{f}^{\mathcal{O}}(t), \tilde{\mathbf{q}}_i) + (\mathbf{g}^{\mathcal{O}}(t), \tilde{\mathbf{q}}_i)_{\Gamma_{\mathrm{N}}}$, and $\tilde{u}_i$ and $\tilde{v}_i$ are the coefficients best fitting $\mathbf{u}^{\mathcal{O}}$ and $\mathbf{v}^{\mathcal{O}}$ in the enhanced eigenvector basis, that is

$$\mathbf{u}^{\mathcal{O}} \approx \sum_{i=1}^{N_{\mathrm{dof}}} \tilde{\mathbf{q}}_i(\mathbf{x}) \tilde{u}_i \quad \text{and} \quad \mathbf{v}^{\mathcal{O}} \approx \sum_{i=1}^{N_{\mathrm{dof}}} \tilde{\mathbf{q}}_i(\mathbf{x}) \tilde{v}_i. \tag{13}$$

Once the approximation $\tilde{\mathbf{u}}^{\mathrm{d}}$ is available, the error in the quantity of interest is assessed using equation (8).

3.2 Assessing timeline-dependent quantities

Recall that, for a given time $t \in I$, $s(t) = L_{\mathrm{TL}}^{\mathcal{O}}(\mathbf{u})(t)$. In that sense, for this particular value of t, $s(t)$ is seen as a scalar quantity of interest taking t as the *final* time. This scalar quantity of interest is characterized as $L^{\mathcal{O}}(\cdot) = L^{\mathcal{O}}_{\mathrm{TL}}(\cdot)(t)$. The associated adjoint problem is analogous to the one presented for the scalar quantity of interest and reads:

$$\rho(\ddot{\mathbf{u}}^{\mathrm{d}}_t - a_1\dot{\mathbf{u}}^{\mathrm{d}}_t) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathrm{d}}_t = -\mathbf{f}^{\mathcal{O}} \quad \text{in } \Omega\times[0,t], \tag{14a}$$

$$\mathbf{u}^{\mathrm{d}}_t = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}\times[0,t], \tag{14b}$$

$$\boldsymbol{\sigma}^{\mathrm{d}}_t\cdot\mathbf{n} = -\mathbf{g}^{\mathcal{O}} \quad \text{on } \Gamma_{\mathrm{N}}\times[0,t], \tag{14c}$$

$$\mathbf{u}^{\mathrm{d}}_t = \mathbf{u}^{\mathcal{O}} \quad \text{at } \Omega\times\{t\}, \tag{14d}$$

$$\dot{\mathbf{u}}^{\mathrm{d}}_t = \mathbf{v}^{\mathcal{O}} \quad \text{at } \Omega\times\{t\}, \tag{14e}$$

with the constitutive law

$$\boldsymbol{\sigma}^{\mathrm{d}}_t := \boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{d}}_t - a_2\dot{\mathbf{u}}^{\mathrm{d}}_t). \tag{15}$$

Note that the solution of this problem is denoted by $\mathbf{u}^{\mathrm{d}}_t$ emphasizing that there is a different solution for each time t. Consequently, equation (14) describes a family of problems, one for each time t.

For a particular instance of time t, the error representation of the timeline-dependent quantity of interest $s^{\mathrm{e}}(t)$ is similar to the standard scalar case but taking the adjoint solution $\mathbf{u}^{\mathrm{d}}_t$ related with the particular value $t \in I$, namely

$$s^{\mathrm{e}}(t) = \hat{R}_t(\mathbf{u}^{\mathrm{d}}_t), \tag{16}$$

where $\hat{R}_t(\mathbf{w}) := L_t(\mathbf{w};t) - B_t(\hat{\mathbf{u}},\mathbf{w})$ and

$$\begin{aligned} B_t(\mathbf{v},\mathbf{w}) &:= \int_0^t (\rho(\ddot{\mathbf{v}}(\tau) + a_1\dot{\mathbf{v}}(\tau)), \dot{\mathbf{w}}(\tau))\ \mathrm{d}\tau + \int_0^t a(\mathbf{v}(\tau) + a_2\dot{\mathbf{v}}(\tau), \dot{\mathbf{w}}(\tau))\ \mathrm{d}\tau \\ &\quad + (\rho\dot{\mathbf{v}}(0^+), \dot{\mathbf{w}}(0^+)) + a(\mathbf{v}(0^+), \mathbf{w}(0^+)), \\ L_t(\mathbf{w}) &:= \int_0^t l(\tau; \dot{\mathbf{w}}(\tau))\ \mathrm{d}\tau + (\rho\mathbf{v}_0, \dot{\mathbf{w}}(0^+)) + a(\mathbf{u}_0, \mathbf{w}(0^+)). \end{aligned}$$

Hence, an estimate for $s^{\mathrm{e}}(t)$ is obtained injecting an enhanced adjoint approximation $\tilde{\mathbf{u}}^{\mathrm{d}}_t$ in equation (16)

$$s^{\mathrm{e}}(t) \approx \hat{R}_t(\tilde{\mathbf{u}}^{\mathrm{d}}_t). \tag{17}$$

Obviously, it is not possible in practice to independently compute the infinite solutions $\tilde{\mathbf{u}}^{\mathrm{d}}_t$ (one for each time $t \in I$) and then using them in equation (16) to assess $s^{\mathrm{e}}(t)$. However, taking $\mathbf{f}^{\mathcal{O}}$ and $\mathbf{g}^{\mathcal{O}}$ constant in time (which accounts for a number of interesting cases), the different functions $\mathbf{u}^{\mathrm{d}}_t$ corresponding to different time instances are all equivalent after a time translation. Thus, if $\mathbf{u}^{\mathrm{d}}_t$ is properly computed for a particular value of t, for instance $t = T$, the general functions $\mathbf{u}^{\mathrm{d}}_t$ for $t \neq T$ are easily recovered as a direct post-process of $\mathbf{u}^{\mathrm{d}}_T$. This fundamental result, shown in the following theorem, is the crucial observation that allows the error estimation technique to be brought to fruition.

Theorem 1 *For a given t, let $\mathbf{u}_t^{\mathrm{d}}$ be the solution of the adjoint problem defined by equations* (14)*. Assume that data $\mathbf{f}^{\mathcal{O}}$ and $\mathbf{g}^{\mathcal{O}}$ in* (4) *are constant in time, i.e. $\mathbf{f}^{\mathcal{O}}(\mathbf{x},t) = \mathbf{f}^{\mathcal{O}}(\mathbf{x})$ and $\mathbf{g}^{\mathcal{O}}(\mathbf{x},t) = \mathbf{g}^{\mathcal{O}}(\mathbf{x})$.*

Then, $\mathbf{u}_t^{\mathrm{d}}$ is related with the adjoint solution associated with the final time T, $\mathbf{u}_T^{\mathrm{d}}$, via the time translation

$$\mathbf{u}_t^{\mathrm{d}}(\tau) = \mathbf{u}_T^{\mathrm{d}}(\tau + T - t). \tag{18}$$

Theorem (1) allows to efficiently recover the family of enhanced approximations $\tilde{\mathbf{u}}_t^{\mathrm{d}}$ from the enhanced approximation $\tilde{\mathbf{u}}_T^{\mathrm{d}}$ as

$$\tilde{\mathbf{u}}_t^{\mathrm{d}}(\tau) = \tilde{\mathbf{u}}_T^{\mathrm{d}}(\tau + T - t). \tag{19}$$

Consequently, the approximation $\tilde{\mathbf{u}}_T^{\mathrm{d}}$ is the base for assessing the error both in the scalar and timeline-dependent quantities, providing in the latter case more meaningful information. The translation (19) is done very efficiently by means of the modal description of $\tilde{\mathbf{u}}_T^{\mathrm{d}}$:

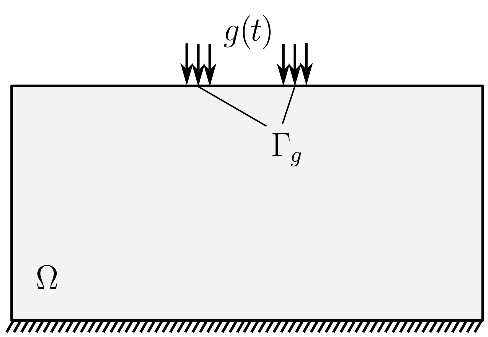
$$\tilde{\mathbf{u}}_t^{\mathrm{d}}(\tau) = \sum_{i=1}^{M} \tilde{\mathbf{q}}_i \tilde{y}_i(\tau + T - t). \tag{20}$$

Recall that, functions $\tilde{y}_i$ may be known analytically in many cases and therefore computing the translation $y_i(\tau + T - t)$ is inexpensive in that cases.

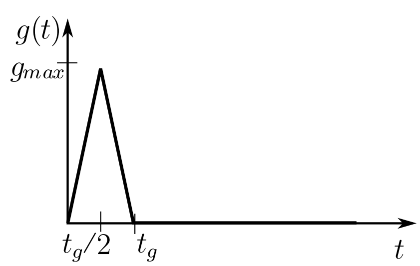
Finally, the error in the timeline-dependent quantity is assessed using the computed adjoint approximations $\tilde{\mathbf{u}}_t^{\mathrm{d}}$ in equation (16).

4 NUMERICAL EXAMPLE

This example illustrates the performance of the proposed error estimates in a 2D wave propagation problem. The problem definition is taken from [9] where it is used to test an error estimate providing error bounds in quantities of interest.



(a) Problem geometry



(b) Time-dependent external load

Figure 2: Example 1: Problem statement.

The problem geometry is the rectangular plate sketched in figure 2(a). The plate is initially at rest ($\mathbf{u}_0 = \mathbf{v}_0 = \mathbf{0}$) and loaded with the time dependent traction

$$\mathbf{g} = \begin{cases} -g(t)\mathbf{e}_2 & \text{on } \Gamma_g, \\ 0 & \text{elsewhere}, \end{cases} \tag{21}$$

where $\mathbf{e}_2 := (0,1)$ and $g(t)$ is the impulsive time-dependent function defined in figure 2(b) with parameters $g_{\max} = 30$ Pa and $t_g = 0.005$ s. No body force is acting in this example ($\mathbf{f} = \mathbf{0}$).

Table 1 details the geometrical parameters and material data, where E and ν are the Young's modulus and Poisson's ratio respectively and the parameter ξ is the dimensionless damping factor. In the examples included her we take $a_1 = 0$, and its corresponding value is $\xi := \frac{1}{2}\omega_1 a_2$, see [9, 11]. Three different values of the viscosity parameter a_2 are considered. The solution of the problem consists of elastic waves propagating along the plate, see [9] for a qualitative description of the solution.

Table 1: Example 1: Problem parameterization

Geometry			Material properties		
Ω	$(-0.5, 0.5) \times (0, 0.5)$	m^2	E	$8/3$	Pa
Γ_g	$[(0.075, 0.125) \cup (-0.075, -0.125)] \times (0.5)$	m	ν	$1/3$	
T	0.25	s	ρ	1	$\mathrm{kg/m}^3$
			a_1	0	s
			a_2	$\{0, 10^{-4}, 10^{-2}\}$	s
			ξ	$\{0, 0.0247, 2.47\}$	%

The timeline-dependent quantity considered in this example is

$$s(t) = (\rho \mathbf{q}_1, \dot{\mathbf{u}}(t)).$$

The quantity s_T is associated with the *exact* first eigenvector of the generalized eigenvalue problem (9) in the Sobolev space $\boldsymbol{\mathcal{V}}_0$. In the following, the unknown function $\mathbf{q}_1$ is replaced by a reference eigenvector $\mathbf{q}_1^{H,p+1}$ solution of the eigenvalue problem (9) in the discrete space $\boldsymbol{\mathcal{V}}_0^{H,p+1}$. The space $\boldsymbol{\mathcal{V}}_0^{H,p+1}$ is obtained increasing by one the interpolation order of $\boldsymbol{\mathcal{V}}_0^{H}$.

Figure 3 shows the reference and approximated timeline quantities $s(t)$ and $\hat{s}(t) := (\rho \mathbf{q}_1, \dot{\hat{\mathbf{u}}}(t))$ and the reference and estimated errors $s^{\mathrm{e}}(t)$ and $\hat{s}^{\mathrm{e}}(t)$ for mesh id. 1 and time step id. 3, see table 2. The proposed estimate $\tilde{s}^{\mathrm{e}}(t)$ is really close to the reference value $s^{\mathrm{e}}(t)$ in all cases, also for $a_2 = 0$. It can be observed that, in this example, the quantity of interest associated to the lowest eigenvector $\mathbf{q}_1$ is nearly unaffected by the change in the damping coefficient a_2. However, the time dependent errors $s^{\mathrm{e}}(t)$ and its approximations $\tilde{s}^{\mathrm{e}}(t)$ are smoothed out as the coefficient a_2 increases.

Table 2: Example 1: Space and time discretizations

Mesh id.	N_{nod}	# Elements	Type	H [m]
1	3051	5899	Triangle	$3.2 \cdot 10^{-3}$
2	12000	23596	”	$1.6 \cdot 10^{-3}$
3	47595	94384	”	$7.9 \cdot 10^{-4}$

Time step id.	# steps	Δt [s]
1	100	$2.5 \cdot 10^{-3}$
2	200	$1.3 \cdot 10^{-3}$
3	400	$6.2 \cdot 10^{-4}$
4	800	$3.1 \cdot 10^{-4}$

5 CONCLUSIONS

This article presents a new type of goal-oriented error estimates assessing the error in timeline-dependent quantities of interest. Timeline-dependent quantities are outputs of the solution describing the time evolution of some space-post-processed functional. Compared to the traditional scalar quantities of interest, this approach fits better the requirements of end-users in dynamic problems. Assessing the error in timeline-dependent quantities involves a family of infinite adjoint problems (one for each time instant in the time interval under consideration). However, all these adjoint problems are similar and they can be recovered from a common parent problem (associated with the a scalar quantity of interest) by means of a simple translation (shift) of the time variable.

The second novelty in this paper is the approximation of the adjoint problem using a decomposition into vibration modes. This allows efficiently precomputing and storing the adjoint solution. Thus, the error estimate is computed along the time integration of the original problem. This approach applies both for the scalar and timeline quantities, but it is specially indicated for the latter because it simplifies the implementation of the time shift.

The error estimation strategies proposed in this work are based on an explicit approach. The error estimate is computed injecting an enhanced approximation of the adjoint solution into the residual of the direct problem. The enhancement is based on a local postprocess of the computed eigenvectors, performed only once and not at each time step. This approach is very efficient for some quantities of interest in which the adjoint solution is fairly represented in a modal description.

The numerical examples show that the proposed estimates have a good effectivity for both the scalar and timeline quantities of interest, accounting both for space and time discretization errors. Contrary to other error estimates for linear visco-elastodynamics, the proposed estimates do not degenerate in the limit case of pure elasticity (i.e. when

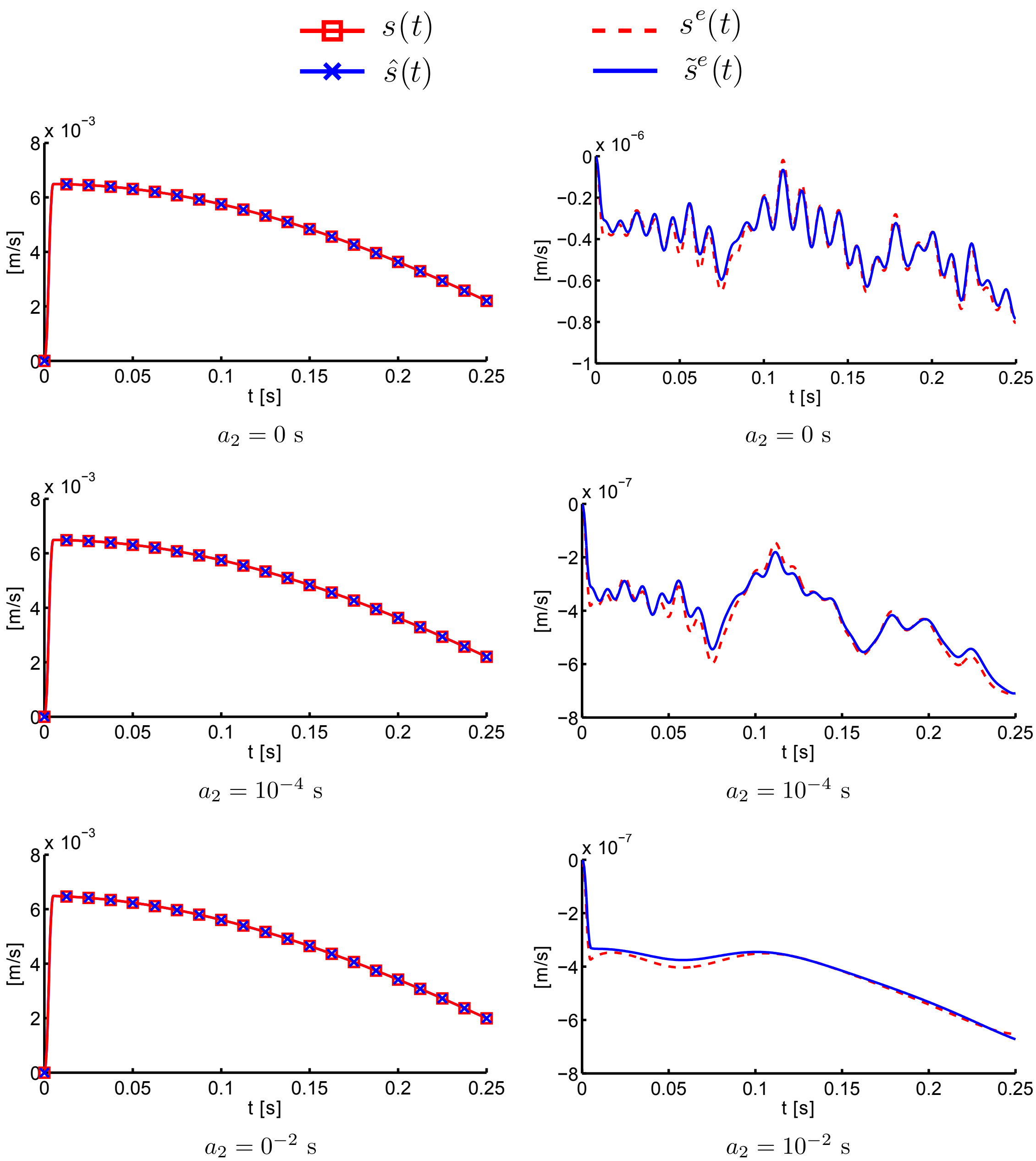


Figure 3: Example 1: Approximated and reference timeline-dependent quantity (left) and estimated and reference errors in the timeline-dependent quantity (right) for the three values of the damping parameter a_2 ($a_2 = 0\,$s, top; $a_2 = 10^{-2}$s, center; $a_2 = 10^{-4}$s, bottom).

no damping is introduced in the formulation).

In current ongoing work, the proposed error estimation techniques are used as driving indicators for mesh adaptivity.

REFERENCES

[1] I Babuŝka and WC Rheinboldt. Error estimates for adaptive finite element computations. *SIAM J. Numer. Anal.*, 18:736–754, 1978.

[2] P Ladevèze and D Leguillon. Error estimate procedure in the finite element method. *SIAM J. on Numerical Analysis*, 20:485–509, 1983.

[3] OC Zienkiewicz and JZ Zhu. A simple error estimator and adaptative procedure for practical engineering analysis. *Int. J. Numer. Meth. Engrg.*, 24:337–357, 1987.

[4] M Paraschivoiu, J Peraire, and AT Patera. A posteriori finite element bounds for linear-functional outputs of elliptic partial differential equations. *Comput. Methods Appl. Mech. Engrg.*, 150:289–321, 1997.

[5] N Parés, J Bonet, A Huerta, and J Peraire. The computation of bounds for linear-functional outputs of weak solutions to the two-dimensional elasicity equations. *Comput. Methods Appl. Mech. Engrg.*, 195:406–429, 2006.

[6] F Cirak and E Ramm. A posteriori error estimation and adaptivity for linear elasticity using the reciprocal theorem. *Comput. Methods Appl. Mech. Engrg.*, 156:351–362, 1998.

[7] S Prudhomme and JT Oden. On goal–oriented error estimation for elliptic problems: application to the control of pointwise errors. *Comput. Methods Appl. Mech. Engrg.*, 176:313–331, 1999.

[8] P Ladevèze and JP Pelle. *La maîtrise du calcul en mécanique linéaire et non linéaire.* Lavoisier, 2001.

[9] F. Verdugo and P. Diez. Computable bounds of functional outputs in linear visco-elastodynamics. *Comput. Methods Appl. Mech. Engrg.*, 245–246:313–330, 2012.

[10] N.E. Wiberg, R. Bausys, and P. Hager. Adaptive h-version eigenfrequency analysis. *Computers and structures*, 71:565–584, 1999.

[11] J Waeytens. *Contrôle des calculs en dynamique: bornes strictes et pertinents sur une quantié d'intérêt.* PhD thesis, LMT-Cachan, 2010.