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A POSTERIORI ERROR ESTIMATES FOR A NEUMANN-NEUMANN DOMAIN DECOMPOSITION ALGORITHM APPLIED TO CONTACT PROBLEMS

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Abstract. In this paper we present an error estimator for unilateral contact problem solved by a Neumann-Neumann domain decomposition algorithm. We also propose errors indicators that take into account the different approximation errors introduced by the computation: the finite element spatial discretisation and the domain decomposition algorithm.

1 INTRODUCTION

Contact problems are frequent in structural analysis. They are characterized by inequality constraints such as non-interpenetration conditions, sign condition on the normal constraints, and an active contact, an area that is a priori unknown. Several approaches exist for solving the non linear equations issued from the finite element discretization of frictionless contact problems. In this work, we consider a natural Neuman-Neumann domain decomposition algorithm, in which each iterative step consists of a Dirichlet problem for the one body, a contact problem for the other one and two Neumann problems to coordinate contact stresses. Two main approximation errors are introduced by this algorithm: a discretization error due to the finite element method (FEM) and an algebraic error due to the Neuman-Neuman domain decomposition algorithm (NNDD).

The objective of this paper is to present an a posteriori global error estimator for a frictionless contact problem, solved by a NNDD algorithm and two errors indicators which

allow to estimate the part of the error due to the spatial discretization and the part of the error due to the domain decomposition algorithm. We show how to extend the error measure in the constitutive relation developed in [4] for contact problems solved by a Neumann-Dirichlet domain decomposition algorithm and how to modify the construction of the admissible fields. The proposed errors estimators and indicators are studied on 2D-examples.

2 CONTACT PROBLEM

We consider the problem of two elastic bodies Ω^1 and Ω^2 in unilateral contact along an interface Γ_c as shown on Figure (1). We choose the orientation of the contact zone Γ_c by setting: $\mathbf{n}^c = \mathbf{n}^1$. Then, we introduce on the interface Γ_C the functions $\mathbf{w}^1$, $\mathbf{w}^2$, representing two displacement fields (one on each side of the interface), $\mathbf{t}^1$, $\mathbf{t}^2$, representing two fields of surface density forces (stresses transmitted to Ω^1 and Ω^2) and $\mathbf{t}^c$ an interior field of surface density forces. The problem of unilateral contact consists of finding $(\mathbf{u}^\alpha, \boldsymbol{\sigma}^\alpha)$ defined on Ω^α $(\alpha = 1, 2)$ and $(\mathbf{w}^1, \mathbf{w}^2, \mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^c)$ defined on Γ_C such that $(\mathbf{u}^\alpha, \mathbf{w}^\alpha, \mathbf{w}^c)$ satisfy the kinematic conditions (1), $(\boldsymbol{\sigma}^\alpha, \mathbf{t}^\alpha, \mathbf{t}^c)$ satisfy the equilibrium equations (2), $(\mathbf{u}^\alpha, \boldsymbol{\sigma}^\alpha)$ satisfy the elastic constitutive relation (3), $(\mathbf{w}^c, \mathbf{t}^c)$ satisfy the contact constitutive relation (4) (for the sake of simplicity we do not consider volumic forces).

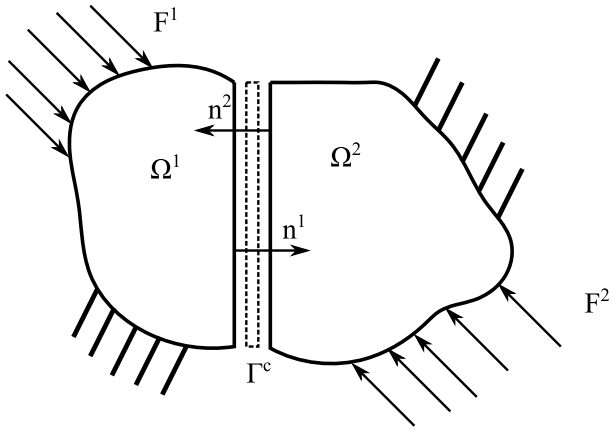

Figure 1: Notations

$$\mathbf{u} \in \mathcal{V}^\alpha, \quad \mathbf{u}^\alpha - \mathbf{w}^\alpha = 0 \text{ and } \mathbf{w}^c = \mathbf{w}^1 - \mathbf{w}^2 \text{ on } \Gamma_c \tag{1}$$

$$\forall\, \mathbf{v} \in \mathcal{V}_0^\alpha, \quad -\int_{\Omega^\alpha} \boldsymbol{\sigma}^\alpha \boldsymbol{\varepsilon}(\mathbf{v})\, dV + \int_{\Gamma_N^\alpha} \mathbf{F}^\alpha \mathbf{v}\, dS + \int_{\Gamma_c} \mathbf{t}^\alpha \mathbf{v}\, dS = 0 \tag{2}$$

$$\mathbf{t}^c - \mathbf{t}^1 = 0 \text{ and } \mathbf{t}^c + \mathbf{t}^2 = 0 \text{ on } \Gamma_c$$

$$\boldsymbol{\sigma}^\alpha = \mathbf{K}^\alpha \boldsymbol{\varepsilon}(\mathbf{u}^\alpha) \text{ in } \Omega^\alpha, \tag{3}$$

$$\phi(-\mathbf{w}^c) + \phi^*(\mathbf{t}^c) + \mathbf{t}^c.\mathbf{w}^c = 0 \text{ on } \Gamma_c, \tag{4}$$

where $\mathcal{V}^\alpha = \{\mathbf{v} \in H^1(\Omega^\alpha);\ \mathbf{u} = \mathbf{u}_D^\alpha \text{ on } \Gamma_D^\alpha\}$, $\mathcal{V}_0^\alpha = \{\mathbf{v} \in H^1(\Omega^\alpha);\ \mathbf{v} = 0 \text{ on } \Gamma_D^\alpha\}$, and for any vector $\mathbf{v}$, the normal and the tangential components are defined according to $v_n = \mathbf{v}.\mathbf{n}^c$ and $v_t = \mathbf{v} - v_n\,\mathbf{n}^c$ and the convex potentials ϕ and ϕ^* are defined by

$$\phi(\mathbf{v}) = \begin{cases} 0 & \text{if } v_n \geq 0 \\ +\infty & \text{otherwise} \end{cases} \quad \text{and } \phi^*(\mathbf{g}) = \begin{cases} 0 & \text{if } g_n \leq 0 \text{ and } g_t = 0 \\ +\infty & \text{otherwise} \end{cases} \tag{5}$$

moreover for any pair (w, t) defined on Γ_c, the Legendre-Fenchel inequality leads to

$$\phi(-\mathbf{w}^c) + \phi^*(\mathbf{t}^c) + \mathbf{t}^c.\mathbf{w}^c \geq 0 \tag{6}$$

Remark: Following [2], the relation defined by equation (4) is equivalent to the Coulomb's constitutive law (7) in a frictionless case.

$$\mathbf{w}_n^c \leq 0, \quad \mathbf{t}_n^c \leq 0, \quad \mathbf{t}_n^c\,\mathbf{w}_n^c = 0 \text{ and } \mathbf{t}_t^c = 0 \text{ on } \Gamma_c, \tag{7}$$

3 DOMAIN DECOMPOSITION ALGORITHM

In this section we briefly recall the Neumann-Neumann domain decomposition algorithm used to solve the unilareral contact problem defined by equations (1)–(4). Given a non-negative parameter θ and an initial arbitrary normal displacement λ_1 defined on Γ_c, we define two sequences of displacements $\mathbf{u}_p^\alpha$ on each solid Ω^α, $\alpha = 1, 2$. Each iteration p of the NNDD algorithm is divided in two successive steps.

- Step 1 – Two independent elasticity problems are solved on Ω_1 and Ω_2:

  1. In Ω^1, the variational problem writes: Find $(\mathbf{u}_p^1, \boldsymbol{\sigma}_p^1)$ defined on Ω^1 and $(\mathbf{w}_p^1, \mathbf{t}_p^1)$ defined on Γ_c such that

$$\mathbf{u}_p^1 = \mathbf{u}_D^1 \text{ on } \Gamma_D^1, \quad \mathbf{u}_p^1 - \mathbf{w}_p^1 = 0 \text{ and } w_p^1\mathbf{n}^1 = \lambda_p \text{ on } \Gamma_c \tag{8}$$

$$\forall\, \mathbf{v} \in \mathcal{V}_0^1, \quad -\int_{\Omega^1} \boldsymbol{\sigma}_p^1 : \boldsymbol{\varepsilon}(\mathbf{v})\, dV + \int_{\Gamma_N^1} \mathbf{F}^1\mathbf{v}\, dS = 0 \tag{9}$$

$$\mathbf{t}_p^1 = \boldsymbol{\sigma}_p^1\mathbf{n}^1 \text{ on } \Gamma_c$$

$$\boldsymbol{\sigma}_p^1 = \mathbf{K}^1\boldsymbol{\varepsilon}(\mathbf{u}_p^1) \text{ in } \Omega^1, \tag{10}$$

  2. In Ω^2, with the given λ_p normal displacement defined on Γ_c, we solve the following variational problem corresponding to a unilateral frictionless contact problem on Γ_c :

$$\mathbf{u}_p^2 = \mathbf{u}_D^2 \text{ on } \Gamma_D^2, \quad \mathbf{u}_p^2 - \mathbf{w}_p^2 = 0 \text{ and } \mathbf{w}_p^c = \lambda_p\mathbf{n}^1 - \mathbf{w}_p^2 \text{ on } \Gamma_c \tag{11}$$

$$\forall\, \mathbf{v} \in \mathcal{V}_0^2, \quad -\int_{\Omega^2} \boldsymbol{\sigma}_p^2 : \boldsymbol{\varepsilon}(\mathbf{v})\, dV + \int_{\Gamma_N^2} \mathbf{F}^2 \mathbf{v}\, dS + \int_{\Gamma_c} \mathbf{t}_p^2 \mathbf{v}\, dS = 0 \tag{12}$$

$$\mathbf{t}_p^c + \mathbf{t}_p^2 = 0 \text{ on } \Gamma_c$$

$$\boldsymbol{\sigma}_p^2 = \mathbf{K}^2 \boldsymbol{\varepsilon}(\mathbf{u}_p^2) \text{ in } \Omega^2, \tag{13}$$

$$\phi(-\mathbf{w}_p^c) + \phi^*(\mathbf{t}_p^c) + \mathbf{t}_p^c.\mathbf{w}_p^c = 0 \text{ on } \Gamma_c, \tag{14}$$

- Step 2 – With $\mathbf{t}_p^1$ and $\mathbf{t}_p^2$ obtained in step 1, we solve two independent "Neumann type" problems:

  In Ω^1, we solve

$$\begin{cases} \text{Find } \mathbf{w}^1 \in \mathbf{V}^1 \text{ such that} \\ -\int_{\Omega^1} \mathbf{K}^1 \boldsymbol{\varepsilon}(\mathbf{w}^1) : \boldsymbol{\varepsilon}(\mathbf{v})\, dV = -\int_{\Gamma_c} \frac{1}{2}(\mathbf{t}_p^1 + \mathbf{t}_p^2).(\mathbf{u}^* - \mathbf{w}^1) \quad \forall \mathbf{u}^* \in \mathbf{V}^1. \end{cases} \tag{15}$$

  In Ω^2, we solve

$$\begin{cases} \text{Find } \mathbf{w}^2 \in \mathbf{V}^2 \text{ such that} \\ -\int_{\Omega^2} \mathbf{K}^2 \boldsymbol{\varepsilon}(\mathbf{w}^2) : \boldsymbol{\varepsilon}(\mathbf{v})\, dV = \int_{\Gamma_c} \frac{1}{2}(\mathbf{t}_p^1 + \mathbf{t}_p^2).(\mathbf{u}^* - \mathbf{w}^2) \quad \forall \mathbf{u}^* \in \mathbf{V}^2. \end{cases} \tag{16}$$

  Let ε_t be the precision of the algorithm, we have the alternative :

  1. If ε_τ is small enough, the algorithm stops.
  2. Else, the normal displacement λ_p is updated :

$$\lambda_{p+1} := \lambda_p + \theta(\mathbf{w}^1 - \mathbf{w}^2).\mathbf{n}$$

     and we return to step 1 for iteration $p+1$.

If $\mathbf{t}_p^1 + \mathbf{t}_p^2 = 0$, it means that the equilibrium is satisfied on the contact interface, in other words the solutions $\mathbf{u}^1$ and $\mathbf{u}^2$ of step 1 constitute the unique solution of the reference problem (1)– (4).

The convergence is obtained when $|\mathbf{w}^1 - \mathbf{w}^2| \longrightarrow 0$. The proof of convergence of the NNDD algorithm (1)-(16) is given in [5] for any sufficiently small $\theta > 0$: There is a $\theta_0 > 0$ such that for any $0 < \theta \leq \theta_0$, the NNDD algorithm for unilateral frictionless contact converges.

At each step p, the approximate solution of problems (8-10) and (11-14) are computed using a classical F.E. method. The finite element spaces are denoted $\mathcal{V}_h^\alpha (\subset \mathcal{V}^\alpha)$ and the approximate solution $(\mathbf{d}_{p,h}, \mathbf{s}_{p,h})$ with $\mathbf{d}_{p,h} = (\mathbf{u}_{p,h}^1, \mathbf{w}_{p,h}^1, \mathbf{u}_{p,h}^2, \mathbf{w}_{p,h}^2, \mathbf{w}_{p,h}^c)$ and $\mathbf{s}_{p,h} = (\boldsymbol{\sigma}_{p,h}^1, \mathbf{t}_{p,h}^1, \boldsymbol{\sigma}_{p,h}^2, \mathbf{t}_{p,h}^2, \mathbf{t}_{p,h}^c)$.

4 ERROR ESTIMATION

4.1 Error in the constitutive relation

To develop an error estimation for a contact problem we use a method based on the constitutive relation error [6]. We recall here the error measure proposed in [1] for a global unilateral contact problem. Let us consider an approximate solution of problem defined by equations (1-4), denoted $(\hat{\mathbf{u}}, \hat{\mathbf{c}})$. The pair $(\hat{\mathbf{d}}, \hat{\mathbf{s}})$ is said to be an *admissible* solution if $(\hat{\mathbf{d}}, \hat{\mathbf{s}}) \in \mathcal{U}_{ad} \times \mathcal{S}_{ad}$ with

- $\mathcal{U}_{ad} = \{\hat{\mathbf{d}} = (\hat{\mathbf{u}}^1, \hat{\mathbf{u}}^2, \hat{\mathbf{w}}^1, \hat{\mathbf{w}}^2, \hat{\mathbf{w}}^c)$ such that $\hat{\mathbf{d}}$ satisfy Eq. (1) and $\phi(-\hat{\mathbf{w}}^c) = 0\}$
- $\mathcal{S}_{ad} = \{\hat{\mathbf{s}} = (\hat{\boldsymbol{\sigma}}^1, \hat{\boldsymbol{\sigma}}^2, \hat{\mathbf{t}}^1, \hat{\mathbf{t}}^2, \hat{\mathbf{t}}^c)$ such that $\hat{\mathbf{s}}$ satisfy Eq. (2) and $\phi^*(\hat{\mathbf{t}}^c) = 0\}$

The constitutive relation error on the pair $(\hat{\mathbf{u}}, \hat{\mathbf{c}})$ is defined by

$$e_{CRE}(\hat{\mathbf{d}}, \hat{\mathbf{s}}) = \left[\sum_{\alpha=1}^{2} \|\hat{\boldsymbol{\sigma}}^\alpha - \mathbf{K}^\alpha \boldsymbol{\varepsilon}(\hat{u}^\alpha)\|^2_{\boldsymbol{\sigma},\Omega^\alpha} + 2 \int_{\Gamma_c} \hat{\mathbf{t}}^c \, \hat{\mathbf{w}}^c \, dS \right]^{1/2}, \tag{17}$$

As an extension of the Prager-Synge theorem it was shown in [1] that

$$e_{CRE}(\hat{\mathbf{d}}, \hat{\mathbf{s}}) \geq \left[\sum_{\alpha=1}^{2} \|\hat{\boldsymbol{\sigma}}^\alpha - \boldsymbol{\sigma}^\alpha\|^2_{\boldsymbol{\sigma},\Omega^\alpha} + \|\hat{\mathbf{u}}^\alpha - \mathbf{u}^\alpha\|^2_{\mathbf{u},\Omega^\alpha} \right]^{1/2} \tag{18}$$

4.2 An a posteriori error estimator for a discretized Neumann-Neumann domain decomposition algorithm

When the formulation of the contact problem is obtained by a domain decomposition method the global error depends not only on the FE discretization error but also on the convergence of the iterative algorithm used (i.e. an algebraic error). Here, we develop an error measure based on the constitutive relation error for an unilateral contact problem solved by a Neumann-Neumann domain decomposition algorithm. Let us introduce new admissible spaces defined at each iteration p of the NNDD algorithm by

- $\mathcal{U}^1_{ad}(\lambda_p) = \{\hat{\mathbf{d}}^1 = (\hat{\mathbf{u}}^1, \hat{\mathbf{w}}^1)$ such that $\hat{\mathbf{d}}^1$ satisfy Eq. (8)$\}$,
- $\mathcal{S}^1_{ad} = \{\hat{\mathbf{s}}^1 = (\hat{\boldsymbol{\sigma}}^1, \hat{\mathbf{t}}^1)$ such that $\hat{\mathbf{s}}^1$ satisfy Eq. (9)$\}$,
- $\mathcal{U}^2_{ad}(\lambda_p) = \{\hat{\mathbf{d}}^2 = (\hat{\mathbf{u}}^2, \hat{\mathbf{w}}^2, \hat{\mathbf{w}}^c)$ such that $\hat{\mathbf{d}}^2$ satisfy Eq. (11) and $\phi(-\hat{\mathbf{w}}^c) = 0\}$,
- $\mathcal{S}^2_{ad}(\lambda_p) = \{\hat{\mathbf{s}}^2 = (\hat{\boldsymbol{\sigma}}^2, \hat{\mathbf{t}}^2, \hat{\mathbf{t}}^c)$ such that $\hat{\mathbf{s}}^2$ satisfy Eq. (12) and $\phi^*(\hat{\mathbf{t}}^c) = 0\}$.

The pair $(\hat{\mathbf{d}}^1_p, \hat{\mathbf{s}}^1_p) \in \mathcal{U}^1_{ad}(\lambda_p) \times \mathcal{S}^1_{ad}$ is the solution of problem (8-10) if

$$e^1_{CRE}(\hat{\mathbf{d}}^1_p, \hat{\boldsymbol{\sigma}}^1_p) = \left[\|\hat{\boldsymbol{\sigma}}^1_p - \mathbf{K}^2 \boldsymbol{\varepsilon}(\hat{\mathbf{u}}^1_p)\|^2_{\boldsymbol{\sigma},\Omega^1} \right]^{1/2} = 0 \tag{19}$$

The pair $(\hat{\mathbf{d}}_p^2, \hat{\boldsymbol{\sigma}}_p^2) \in \mathcal{U}_{ad}^2(\lambda_p) \times \mathcal{S}_{ad}^2(\lambda_p)$ is the solution of problem (11-14) if

$$e_{CRE}^2(\hat{\mathbf{d}}_p^2, \hat{\mathbf{s}}_p^2) = \left[\|\hat{\boldsymbol{\sigma}}_p^2 - \mathbf{K}^2 \boldsymbol{\varepsilon}(\hat{\mathbf{u}}_p^2)\|_{\boldsymbol{\sigma},\Omega^2}^2 + 2 \int_{\Gamma_c} \hat{\mathbf{t}}_p^c \, \hat{\mathbf{w}}_p^c \, dS \right]^{1/2} = 0 \tag{20}$$

However, the pair $(\hat{\mathbf{d}}_p = (\hat{\mathbf{d}}_p^1, \hat{\mathbf{d}}_p^2), \hat{\mathbf{s}}_p = (\hat{\mathbf{s}}_p^1, \hat{\mathbf{s}}_p^2))$, is not an admissible solution for the unilateral contact problem (i.e. $\notin \mathcal{U}_{ad} \times \mathcal{S}_{ad}$) because the equilibrium equation (2) is not necessarily satisfied as $\hat{\mathbf{t}}_p^c - \hat{\mathbf{t}}_p^1 = 0$ has not been imposed. The quantity e_{CRE} defined by equation (21) is an error estimator for the problem defined by equations (8-14), when

$$e_{CRE}(\hat{\mathbf{d}}_p^1, \hat{\mathbf{s}}_p^1, \hat{\mathbf{d}}_p^2, \hat{\mathbf{s}}_p^2) = \left[\left(e_{CRE}^1(\hat{\mathbf{d}}_p^1, \hat{\mathbf{s}}_p^1) \right)^2 + \left(e_{CRE}^2(\hat{\mathbf{d}}_p^2, \hat{\mathbf{s}}_p^2) \right)^2 \right]^{\frac{1}{2}} \tag{21}$$

We have the following property

$$e_{CRE}(\hat{\mathbf{d}}_p^1, \hat{\mathbf{s}}_p^1, \hat{\mathbf{d}}_p^2, \hat{\mathbf{s}}_p^2) = 0 \Leftrightarrow (\hat{\mathbf{d}}_p^1, \hat{\mathbf{s}}_p^1, \hat{\mathbf{d}}_p^2, \hat{\mathbf{s}}_p^2) \text{ is the exact solution of (8-14) for a fixed } \lambda_p.$$

This error measure quantify the error due to the Finite Element discretization at each step of the algorithm. In order to obtain a global error estimator for the contact problem, let us define an admissible solution for the unilateral contact problem $(\hat{\hat{\mathbf{d}}}_p = (\hat{\hat{\mathbf{d}}}_p^1, \hat{\hat{\mathbf{d}}}_p^2), \hat{\hat{\mathbf{s}}}_p = (\hat{\hat{\mathbf{s}}}_p^1, \hat{\hat{\mathbf{s}}}_p^2))$ such that

$$(\hat{\hat{\mathbf{d}}}_p^1, \hat{\hat{\mathbf{s}}}_p^1) \in \mathcal{U}_{ad}^1(\lambda_p) \times \mathcal{S}_{ad}^1, \quad (\hat{\hat{\mathbf{d}}}_p^2, \hat{\hat{\mathbf{s}}}_p^2) \in \mathcal{U}_{ad}^2(\lambda_p) \times \mathcal{S}_{ad}^2(\lambda_p) \text{ and } \hat{\hat{\mathbf{t}}}_p^c - \hat{\hat{\mathbf{t}}}_p^1 = 0$$

The global error estimator for the contact problem is defined by

$$\eta^{glo} = e_{CRE}(\hat{\hat{\mathbf{d}}}_p^1, \hat{\hat{\mathbf{s}}}_p^1, \hat{\hat{\mathbf{d}}}_p^2, \hat{\hat{\mathbf{s}}}_p^2) = \left[\left(e_{CRE}^1(\hat{\hat{\mathbf{d}}}_p^1, \hat{\hat{\mathbf{s}}}_p^1) \right)^2 + \left(e_{CRE}^2(\hat{\hat{\mathbf{d}}}_p^2, \hat{\hat{\mathbf{s}}}_p^2) \right)^2 \right]^{\frac{1}{2}} \tag{22}$$

We have the following property

$e_{CRE}(\hat{\hat{\mathbf{d}}}_p^1, \hat{\hat{\mathbf{s}}}_p^1, \hat{\hat{\mathbf{d}}}_p^2, \hat{\hat{\mathbf{s}}}_p^2) = 0 \Leftrightarrow (\hat{\hat{\mathbf{d}}}_p^1, \hat{\hat{\mathbf{s}}}_p^1, \hat{\hat{\mathbf{d}}}_p^2, \hat{\hat{\mathbf{s}}}_p^2)$ is the exact solution of the global unilateral contact problem defi

The admissible displacement fields are easily recovered, since the finite element fields satisfy the kinematic constraints and $\phi(\mathbf{w}_{p,h}^c) = 0$.

$$\hat{\hat{\mathbf{d}}}_p^1 = (\mathbf{u}_{p,h}^1, \mathbf{w}_{p,h}^1) \text{ and } \hat{\hat{\mathbf{d}}}_p^2 = (\mathbf{u}_{p,h}^2, \mathbf{w}_{p,h}^2, \mathbf{w}_{p,h}^c)$$

However, the stress fields and the traction forces $\mathbf{s}_{p,h}$ computed by the algorithm do not satisfy the equilibrium equations. The pair $(\hat{\hat{\mathbf{s}}}_p^1, \hat{\hat{\mathbf{s}}}_p^2)$ is recovered from the finite element solution and the data in 3 steps

- The first step, consist in recovering admissible traction fields $(\hat{\hat{\mathbf{t}}}^1, \hat{\hat{\mathbf{t}}}^2, \hat{\hat{\mathbf{t}}}^c)$. We built a traction $\hat{\hat{\mathbf{t}}}^c$ such that $\phi^*(\hat{\hat{\mathbf{t}}}^c) = 0$ and which minimize in the least square sense $J(\hat{\hat{\mathbf{t}}}^c)$

$$J(\hat{\hat{\mathbf{t}}}^c) = \int_{\Gamma_c} \left(\hat{\hat{\mathbf{t}}}^c - \frac{1}{2} \left(\mathbf{t}^1_{h,p} - \mathbf{t}^2_{h,p} \right) \right)^2 dS$$

- The second step, consist in recovering stress fields $\tilde{\boldsymbol{\sigma}}^\alpha_{h,p}$ that satify the FE-equilibrium equations on each solid Ω^α. Let $\tilde{\mathbf{u}}^\alpha_{h,p} \in \mathcal{V}^\alpha_h$ such that $\tilde{\boldsymbol{\sigma}}^\alpha_{h,p} = \mathbf{K}^\alpha \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}^\alpha_{h,p})$ and

$$\forall\, \mathbf{v} \in \mathcal{V}^\alpha_{h,0}, \quad - \int_{\Omega^\alpha} \mathbf{K}^\alpha \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}^\alpha_{h,p}) : \boldsymbol{\varepsilon}(\mathbf{v})\, dV + \int_{\Gamma^\alpha_N} \mathbf{F}^\alpha \mathbf{v}\, dS + \int_{\Gamma_c} \hat{\hat{\mathbf{t}}}^\alpha \mathbf{v}\, dS = 0$$

- The recovery of equilibrated stress fields $\hat{\hat{\boldsymbol{\sigma}}}^\alpha$ from $\tilde{\boldsymbol{\sigma}}^\alpha_{h,p}$ in each subdomain Ω^α is the most technical point. This step is performed with a traction-free recovery technique developed in [4].

4.3 Error indicators for the NNDD algorithm and for the FE discretization

Following the method proposed in [7, 8], we propose here two error indicators that allow us to estimate separately the part of the error due to the FE discretization from the part due to the NNDD algorithm. The discretization error is defined as the limit of the global error when the convergence criterion of the iterative algorithm tends to zero. The NNDD algorithm error is defined as the limit of the global error as the mesh size h tends to zero.

To define FE discretization error indicator η^{FE}, let us consider the reference problem defined by the step p of the NNDD algorithm: Find $\mathbf{d}_p = (\mathbf{u}^1_p, \mathbf{w}^1_p, \mathbf{u}^2_p, \mathbf{w}^2_p, \mathbf{w}^c_p)$ and $\mathbf{s}_p = (\boldsymbol{\sigma}^1_p, \mathbf{t}^1_p, \boldsymbol{\sigma}^2_p, \mathbf{t}^2_p, \mathbf{t}^c_p)$ that satisfy equations (8-14). The only approximation introduced between $(\mathbf{d}_p, \mathbf{s}_p)$ and the finite element solution $(\mathbf{d}_{p,h}, \mathbf{s}_{p,h})$ is the FE discretization. We have shown in section (4.2) that the error in the constitutive relation $e_{CRE}(\hat{\mathbf{d}}^1_p, \hat{\mathbf{s}}^1_p, \hat{\mathbf{d}}^2_p, \hat{\mathbf{s}}^2_p)$ defined in equation (21) is an error estimator for this reference problem. The quantity $e_{CRE}(\hat{\mathbf{d}}^1_p, \hat{\mathbf{s}}^1_p, \hat{\mathbf{d}}^2_p, \hat{\mathbf{s}}^2_p)$ is used to define a FE discretization error indicator for the unilateral contact problem

$$\eta^{FE} = e_{CRE}(\hat{\mathbf{d}}^1_p, \hat{\mathbf{s}}^1_p, \hat{\mathbf{d}}^2_p, \hat{\mathbf{s}}^2_p) \tag{23}$$

To define NNDD algorithm error indicator η^{NNDD}, let us consider the reference problem defined by finite element discretization of the unilateral contact problem (Eq. (1)–(4)). The only approximation introduced between the solution of this discretized problem and the finite element solution $(\mathbf{d}_{p,h}, \mathbf{s}_{p,h})$ is the approximation introduced by the NNDD algorithm. Let $(\hat{\mathbf{d}}_h, \hat{\mathbf{s}}_h)$ and admissible solution for this problem (i.e. that satisfy the discretized version of Eq. (1)–(2)). The error in the constitutive relation $e_{CRE}(\hat{\mathbf{d}}_h, \hat{\mathbf{s}}_h)$ defined by equation (17) is an error estimator for this reference problem, and is used to define NNDD algorithm error indicator for the unilateral contact problem

$$\eta^{NNDD} = e_{CRE}(\hat{\mathbf{d}}_h, \hat{\mathbf{s}}_h) \tag{24}$$

5 Numerical results

The considered example is shown on figure (2). The lower boundary of structure Ω_1 is clamped, on structure Ω_2 the applied force F^2 has a linear distribution ($F^2_{max} = 10^7$) and the applied displacement is $u^2_D = -10^{-4}$. The Young's modulus for both structures is $E = 210\,GPa$ and the Poisson's ratio is $\nu = 0.27$. The coefficient θ of the NNDD algorithm is set to 0.25.

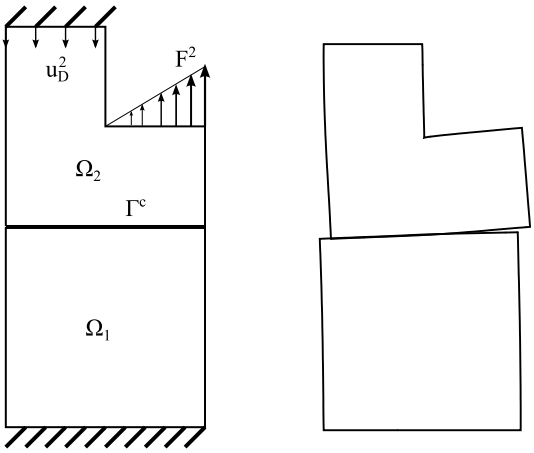

Figure 2: Unilateral contact reference model (left) - distorted structures (right)

We first study the evolution of the global error estimator η^{glo}, of the FE error indicator η^{FE}, and of the NNDD error indicator η^{NNDD} as a function of the number n_{DoF} of the degree of freedom (DoF), for a fixed number iterations of the NDDD algorithm $n_{ite} = 6$. The results are presented on figure (3). The global error η^{glo} tends to an horizontal asymptote which is the NNDD error indicator η^{NNDD}, whereas the convergence of FE error indicator η^{FE} as a function of the number of DoF is shown. The η^{glo} can be numerically related to η^{NNDD} and η^{FE} by relation (25).

$$\left(\eta^{glo}\right)^2 \approx \left(\eta^{FE}\right)^2 + \left(\eta^{NNDD}\right)^2 \tag{25}$$

To evaluate the computed global error, we compute a reference solution denoted by $\mathbf{u}_{ref}$ and we define the reference error e_{ref} and the effectivity index γ by

$$e_{ref} = \left[\sum_{\alpha=1}^{2} \|\mathbf{u}^\alpha_{ref} - \mathbf{u}^\alpha_{h,p}\|^2_{\mathbf{u},\Omega^\alpha}\right]^{1/2} \quad \text{and} \quad \gamma = \frac{\eta^{glo}}{e_{ref}} \tag{26}$$

To obtain a reliable reference solution we choose a mesh size $h_{ref} = 1/8h$ and we set the convergence criteria of the NNDD algorithm to 10^{-8}. The results are reported on figure (4). We first study the evolution of the global error estimator η^{glo}, of the FE error

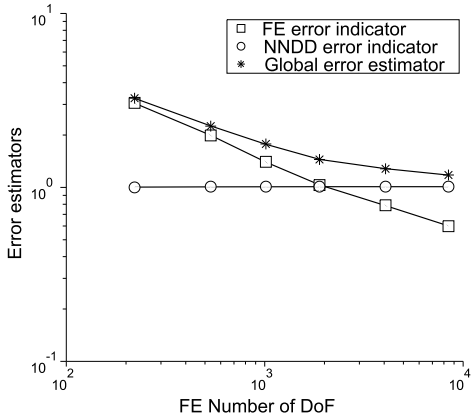

Figure 3: Computed errors as a function of the number of DoF

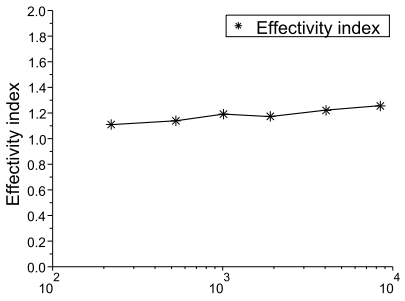

Figure 4: Effectivity index as a function of the number of DoF

indicator η^{FE}, and of the NNDD error indicator η^{NNDD} as a function of the number of n_{ite} the number of iterations of the NDDD algorithm for a fixed number of DoF $n_{DoF} = 1002$. The results are presented on figure (5). The global error η^{glo} tends to an horizontal asymptote which is the FE error indicator η^{FE}, whereas the convergence of NNDD error indicator η^{NDDD} as a function of the number of iterations is shown.

6 CONCLUSION

A global error estimator based on the constitutive relation has been introduced to verify an approximate computation of an unilateral contact problem based on a Finite Element discretization associated with a Neumann-Neumann domain decomposition algorithm. This global error is an upper bound of the exact error and the effectivity index is 1.2 on the studied examples. This error measure takes into account all the errors due to discretization, i.e. both the errors due to the spatial discretization and those due to the domain decomposition algorithm. Two error indicators are developed to estimate the

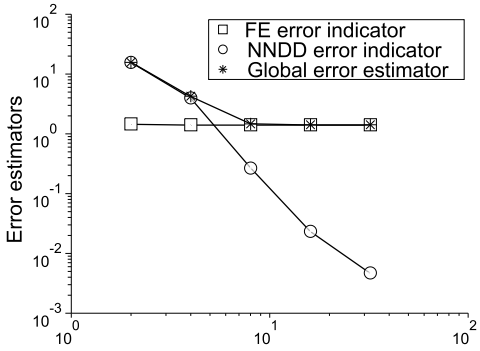

Figure 5: Computed errors as a function of the number iterations of the NNDD algorithm

contributions of each source of error. They are defined in the same way as the error, except that the reference problem is different. On the first tests, these indicators seem to behave well.

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