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NON INTRUSIVE STOCHASTIC SIMULATIONS USING A GOAL ORIENTED ADAPTIVE STRATEGY.

Eric FLORENTIN[*], **Pedro DIEZ**[†]

[*]LMT-Cachan (ENS Cachan / CNRS / UPMC / PRES UniverSud Paris)
61, avenue du Président Wilson, F-94235 Cachan, France
e-mail: eric.florentin@ens-cachan.fr

[†]Laboratori de Càlcul Numric (LaCàN),
Universitat Politècnica de Catalunya, C2 Campus Nord UPC, E-08034 Barcelona, Spain
e-mail: pedro.diez@upc.edu

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Abstract. The paper presents a goal-oriented strategy in the framework of stochastic non-intrusive Monte Carlo finite element simulations. The methods consists in a successive enrichment of a reduced basis. This enrichment is performed on the fly, during the Monte Carlo process. The error made by the representation on the reduced basis is assessed introducing a dual problem associated to the quantity of interest. The efficiency of the proposed approach is illustrated in numerical examples. In particular, an extension of the work developed in [6] is introduced. It consists in introducing a reduced basis for solving the dual problem in an efficient way. Different variant are tested for the successive enrichment of the dual reduced basis.

1 INTRODUCTION

Stochastic Finite Element Methods (FEM) are currently an essential tool for the quantitative prediction of the response of mechanical models that include randomness. Both for the research and industrial players, a key issue is to reduce the computational cost in order to afford dealing with large scale applications. A state of the art for stochastic methods can be found for example in [1, 2, 3].

In [4, 5], the authors introduced a reduced basis methodology to reduce the cost of Monte Carlo simulations, offering an attractive framework for solving stochastic problems with a large number of parameters. The idea is simple and effective because the different Monte Carlo shots lead to similar FE problems and therefore the reduced basis approach is highly performant.

This paper describes a new approach to generate a reduced basis in the context of Monte Carlo strategies for stochastic modeling. The reduced basis is constructed automatically, ensuring a prescribed level of accuracy for the output of interest. The proposed methodology uses standard elements in goal-oriented error assessment and adaptivity [6],. Here, the error which is assessed is the approximation introduced by the Reduced Basis with respect to the complete FE solution, for a given mesh. That differs from the usual practice in the Verification framework, in which the error introduced by the mesh is evaluated. Numerical tests demonstrate the efficiency and robustness of the proposed strategy. The use of this method for 3D massive industrial examples, where the cost is of primary importance, is going to be the object of further research.

2 PROBLEM STATEMENT

Let Ω be a bounded domain and $\partial\Omega$ its boundary which is divided in two parts $\partial_D\Omega$ and $\partial_N\Omega$ such that $\overline{\partial_D\Omega} \cup \overline{\partial_N\Omega} = \partial\Omega$, $\partial_D\Omega \neq \varnothing$ and $\partial_D\Omega \cap \partial_N\Omega = \varnothing$. Displacement $\mathbf{u}_d$ is imposed on $\partial_D\Omega$ and a traction $\mathbf{g}_d$ is applied on $\partial_N\Omega$ and a body force field $\mathbf{f}_d$ is applied in Ω.

The material is linear elastic, and $\mathbf{K}(\mathbf{x},\theta)$ is the Hooke tensor random field, where $\mathbf{x} \in \Omega$ is the position and $\theta \in \Theta$ denotes the randomness. Θ is the set of possible outcomes of θ.

The problem reads: find the unknown displacement field $\mathbf{u}(\mathbf{x},\theta)$ such that

$$\mathbf{div}\left(\mathbf{K}(\mathbf{x},\theta)\varepsilon\left[\mathbf{u}(\mathbf{x},\theta)\right]\right) + \mathbf{f}_d(\mathbf{x},\theta) = \mathbf{0} \qquad \text{in } \Omega \tag{1a}$$

$$\mathbf{K}(\mathbf{x},\theta)\varepsilon\left[\mathbf{u}(\mathbf{x},\theta)\right].\mathbf{n} = \mathbf{g}_d(\mathbf{x},\theta) \qquad \text{on } \partial_N\Omega \tag{1b}$$

$$\mathbf{u}(\mathbf{x},\theta) = \mathbf{u}_d(\mathbf{x},\theta) \qquad \text{on } \partial_D\Omega \tag{1c}$$

The corresponding standard weak form reads as follows: find $\mathbf{u}(\mathbf{x},\theta)$ such that

$$a(\mathbf{u}(\mathbf{x},\theta),\mathbf{w}(\mathbf{x})) = \ell(\mathbf{w}(\mathbf{x})) \quad \forall \mathbf{w}(\mathbf{x}) \in \mathcal{U} \tag{2}$$

where $a(\cdot,\cdot)$ is a bilinear form , $\ell(\cdot)$ is a linear form and $\mathcal{U}$ the set of admissible displacements, satisfying (1c).

3 MONTE CARLO NON INTRUSIVE SOLVING SCHEME

The non-intrusive approach decouples the discretization of the physical space and the stochastic space, represented here by Ω and Θ. This can be described in two steps.

□ **Step 1:** finite element discretization.

In this first phase, the problem is considered as deterministic (*for a given value of* θ.) The discretisation of the space is characterized by the standard finite element functions $N_i(\mathbf{x})$, $i = 1, 2 \ldots N_{FE}$ $\mathcal{U}_h \subset \mathcal{U}$

$$\mathcal{U}_h = \text{span}\left\{N_1, N_2, \ldots N_{N_{FE}}\right\} \tag{3}$$

The numerical approximation in the space defined in (3) is $\mathbf{u}_h$ such that

$$\mathbf{u}(\mathbf{x},\theta) \approx \mathbf{u}_h(\mathbf{x},\theta) = \sum_{i=1}^{N_{FE}} u_i(\theta) N_i(\mathbf{x}) = \mathbf{N}^T(\mathbf{x})\mathbf{U}(\theta), \tag{4}$$

with $\mathbf{U} = [u_1\, u_2\, \cdots\, u_{N_{FE}}]^T$ and $\mathbf{N}(\mathbf{x}) = [N_1(\mathbf{x})\, N_2(\mathbf{x})\, \cdots\, N_{N_{FE}}(\mathbf{x})]^T$. The corresponding discretized form of (2) is the linear system of equations

$$\mathbb{K}(\theta)\mathbf{U}(\theta) = \mathbf{F}(\theta), \tag{5}$$

where $\mathbb{K}$ is the classical finite element stiffness matrix and classical $\mathbf{F}$ nodal forces.

□ **Step 2:** Monte Carlo simulation.

The Monte Carlo technique consists in generating a number N_{MC} of realizations of θ. Note that these realizations are generated using the actual Probability Density Function (PDF) of θ. This is equivalent to determine N_{MC} realizations of $\mathbb{K}(\theta)$ and therefore, solving N_{MC} linear systems of equations (5), obtaining N_{MC} realizations of $\mathbf{U}(\theta)$.

Thus, the PDF of $\mathbf{U}(\theta)$ or some specific Quantity of Interest (linearly dependent with $\mathbf{U}$) is approximated from these realizations.

This non-intrusive strategy is extremely simple because it decouples the approximation of the stochastic behavior and the solution of the deterministic mechanical model. The main drawback is that the numerical cost is can be very large.

4 METHOD DEVELOPPED

4.1 Reduced Basis

The reduced basis method allows to reduce computational costs when solving a large number of problems, but introduce an error linked with the size of the reduced basis size. Monte Carlo sampling requires solving many instances of problem (5) and Reduced Basis Strategy can help. A collection of N_{RB} linearly independent solutions creates a approximated subset of the FE space : $\left\{u_{h(1)}, u_{h(2)}, \ldots, u_{h(N_{RB})}\right\}$ described by the corresponding vectors of nodal values $\{\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{N_{RB}}\}$,

$$\mathcal{U}_{RB} := \text{span}\left\{u_{h(1)}, u_{h(2)}, \ldots, u_{h(N_{RB})}\right\} \subset \mathcal{U}_h \tag{6}$$

Then, the solution of a new instance of (5) is seek in $\mathcal{U}_{RB}$ instead of in $\mathcal{U}_h$ as a linear combination of the elements of the reduced basis:

$$\mathbf{U}_{RB} = \sum_{i=1}^{N_{RB}} a_i \mathbf{U}_i = \mathbb{U}_{RB}\, \mathbf{a} \tag{7}$$

where the matrix $\mathbb{U}_{RB} = [\mathbf{U}_1\, \mathbf{U}_2 \cdots \mathbf{U}_{N_{RB}}]$ (with N_{FE} rows and N_{RB} columns) describes the change of basis and $\mathbf{a}^T = [a_1\, a_2 \cdots a_{N_{RB}}]$ is the vector of unknowns.

The solution $\mathbf{a}$ can be found as the solution of a linear system :

$$\left(\mathbb{U}_{RB}^T \mathbb{K}(\theta) \mathbb{U}_{RB}\right) \mathbf{a} = \left(\mathbb{U}_{RB}^T \mathbf{F}(\theta)\right) \text{ readily rewritten as } \mathbb{K}_{RB}(\theta)\mathbf{a}(\theta) = \mathbf{F}_{RB}(\theta) \tag{8}$$

4.2 Goal oriented error

At every instance θ_k of the Monte Carlo process, the error introduced in the reduced basis phase is measured by

$$\mathbf{E}_{RB} := \mathbf{U}_{RB} - \mathbf{U}, \tag{9}$$

This error vector, expressed in the standard FE basis, corresponds to the approximation introduced by the reduced basis with respect to the complete FE solution. It does not represent the error due to the Finite Element discretization, but the error uniquely do to the reduced basis process.

The squared norm of the residual, $\mathbf{R}_{RB}^T\mathbf{R}_{RB}$ is in fact a good error indicator for the energy norm of the error with

$$\mathbf{R}_{RB} := \mathbb{K}\mathbf{U}_{RB} - \mathbf{F} \tag{10}$$

If interested by a QoI denoted Q_{RB} associated with the reduced basis solution $\mathbf{U}_{RB}$,

$$Q_{RB} = \mathbf{G}^T\mathbf{U}_{RB} \tag{11}$$

it is classical to introduce a dual problem and its solution $\mathbf{V}$ and where $\mathbf{G}$ defines the quantity of interest. Using the same spatial discretisation for dual problem as the direct problem (5), it results a similar system for $\mathbf{V}$

$$\mathbb{K}(\theta)\mathbf{V}(\theta) = \mathbf{G}(\theta), \tag{12}$$

This dual solution $\mathbf{V}$ can be used to assess the error in the evaluation of the QoI associated with the reduced basis. The error in the QoI associated with the reduced basis approach writes:

$$e_{RB}^Q := \mathbf{V}^T\mathbf{R}_{RB}. \tag{13}$$

5 Adaptive Strategies

5.1 Algorithms

Note that if $\mathbf{V}$ is known, the error in the QoI associated with the reduced basis is computed explicitly using the right-hand side term of (13), once the reduced basis solution is available. The problem is that solving (12) leads to a similar computational cost that for solving 5. The local error 13 can be used to control the enrichment of an adapted reduced basis. Indeed, this allows assessing the error committed in every shot with a low computational cost and decide on the fly if the reduced basis is rich enough or if it has to be enriched further.

Two adaptation strategies, corresponding to different level of randomness are proposed.

□ **Algo 1:** Dual unique resolution

From the practical viewpoint it is assumed that the variation of $\mathbf{V}$ with the randomness is small in such a way that $\mathbf{V}$ is kept constant in order to estimate the error in the QoI.

Thus, the solution of the dual system (12) is performed just once and leads to $\mathbf{V_0}$. The criterion used to estimate error is then:

$$e^Q_{RB1} = \mathbf{V_0}^T \mathbf{R}_{RB} \tag{14}$$

This algorithm is presented in [6].

□ **Algo 2:** Dual Reduced Basis Resolution

For problem with higher level of randomness, the solution $\mathbf{V_0}$ can be very different of the actual value $\mathbf{V}$. Then, the idea correspond to a resolution of the dual problem using a reduced basis representation.

$$\mathbf{V}_{RB} = \sum_{i=1}^{N_{RBD}} b_i \mathbf{V}_i = \mathbb{V}_{RB}\, \mathbf{b} \tag{15}$$

where the matrix $\mathbb{V}_{RB} = [\mathbf{V}_1\, \mathbf{V}_2 \cdots \mathbf{V}_{N_{RB}}]$ (with N_{FE} rows and N_{RBD} columns) describes the change of basis and $\mathbf{b}^T = [b_1\, b_2 \cdots B_{N_{RBD}}]$ is the vector of unknowns.

Then the solution $\mathbf{b}$ can be found as the solution of a linear system :

$$\left(\mathbb{V}^T_{RB}\mathbb{K}(\theta)\mathbb{V}_{RB}\right)\mathbf{b} = \left(\mathbb{V}^T_{RB}\mathbf{G}(\theta)\right) \text{ readily rewritten as } \mathbb{K}_{RB}(\theta)\mathbf{b}(\theta) = \mathbf{G}_{RB}(\theta) \tag{16}$$

The criterion used to estimate error is then:

$$e^Q_{RB2} = (\mathbb{V}_{RB}\, \mathbf{b})^T \mathbf{R}_{RB} \tag{17}$$

Remark 1: The choice of a unique vector in the basis $\mathbb{V}_{RB} = V_0$ correspond exactly to algorithm 1. In this sense, this resolution correspond to a generalization of the algorithm 1.

Remark 2: A particular case correspond to the same reduced basis to obtain the approximated solution of both primal an dual problem.

$$\mathbb{U}_{RB} = \mathbb{V}_{RB} \tag{18}$$

Remark 3: Different variants that correspond to the decision to enrich independently $\mathbb{U}_{RB}$ and $\mathbb{V}_{RB}$ or not can be performed. The decision to enrich $\mathbb{V}_{RB}$ can be based on a dual error criterion :

$$e^{QD}_{RB2} = (\mathbb{U}_{RB}\, \mathbf{a})^T (\mathbb{K}\mathbf{V}_{RB} - \mathbf{G}) \tag{19}$$

5.2 Results

Results of algorithm 1 and algorithm 2 are compared to a full Monte Carlo simulation which is considered as the reference. In [6] the results illustrates the capabilities of algorithm 1 for a 2D mechanical problem. This work focuses on the different possibilities to solve the problem using algorithm 2 which is an extension of the work presented in [6] and the results illustrate the algorithm 2 developed here. Results are given in Table 1,

Table 1: Results for algorithm 2

ϵ_0	N_{RB}	N_{RB}	ϵ
10^2	1	1	1.3E+1
10	8	6	2.4E+0
1	18	16	2.0E-1
10^{-1}	38	38	1.9E-2
10^{-2}	93	79	2.5E-3
10^{-3}	139	122	2.2E-4

for a given level of prescribed local error ϵ_0 on the QoI, the actual error ϵ is given and is to be compared to the prescribed one ϵ_0. The size of the Reduced basis N_{RB} and N_{RBD} is also given. The results are showing that the algorithm makes it possible to respect a given level of error on the quantity of interest. The employed reduced basis for solving primal and dual problems are automatically adapted by enrichment on the fly.

6 CONCLUSIONS

Next step is to implement this algorithm and its variant on 3D large scale examples where the problem of computational costs are of primary importance.

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