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ROBUST ERROR ESTIMATES IN WEAK NORMS WITH APPLICATION TO IMPLICIT LARGE EDDY SIMULATION

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Abstract. We discuss a posteriori and a priori error estimates of filtered quantities for solutions to some equations of fluid mechanics. For the computation of the solution we use low order finite element methods with either linear or nonlinear stabilization. The aim is to make the constants of the estimates independent of the Reynolds number, the Sobolev norm of the exact solution at time $t > 0$, or nonlinear effects such as shock formation. For the case of Burgers' equation this is possible. It follows that we obtain a complete assessment of the computability of the solution given the initial data. After a detailed description of the results in the case of the Burgers' equation we widen the scope and discuss transient convection–diffusion equations with rough data and the incompressible Navier-Stokes' equations in two space dimensions within the same paradigm.

1 INTRODUCTION

The task of designing adaptive finite element methods for flow problems reamains a challenging problem. A major bottleneck is the need to find a posteriori error estimators that are robust with respect to the Reynolds/Péclet number. In engineering practice a popular approach has been to use dual weighted residual type estimates in order to capture the stability properties of the problem at hand by solving a dual problem. This methodology however lacks theoretical underpinning, indicating when the approach is likely to work or to fail, in particular in the convection dominated regime. The aim of the present paper is to present some basic results showing that in the one dimensional case, or for special scale separated solutions in two space dimensions, robustness can be obtained for estimates of filtered quantities, provided a stabilized finite element method is used. We will here give an overview of recent results. For full proofs of the given results we refer to the recent publications [2, 3, 4].

For the discretization we use finite element methods with piecewise affine continuous approximation and linear, or nonlinear, artificial viscosity or higher order symmetric stabilization. These methods are strongly related to so called implicit large eddy simulation(ILES) methods where turbulent flows are approximated using the Navier-Stokes' equations and a discretization scheme augmented with some dissipative operator to guarantuee numerical stability, see [1].

In this framework we prove estimates for a regularized error. The interest of these estimates stems from the fact that the constant of the estimates are of moderate size and only depends on the regularity of the initial data in one space dimension, and in several dimensions the gradients of the large, energy carrying vortices. Hence there is no dependence on the Reynolds number, nor of the global regularity of the exact solution. The estimates also give a precise rate of convergence in the meshsize h, depending only on the filter width. This can be seen as a tentative theoretical explanation to the good performance of ILES methods for two dimensional flows in the absence of backscatter effects [7]. In this context our scale separation assumption (Assumption 1) acts as a sufficient condition to eliminate backscatter.

We will consider the following differential filter that sometimes is applied as a regularization in modified Navier-Stokes' systems for large eddy simulation,

$$-\delta^2 \Delta \tilde{u} + \tilde{u} = u(\cdot, T) \quad \text{on } \Omega \tag{1}$$

with $\tilde{u} = 0$ on $\partial\Omega$ and δ denoting the filter width. Let $\tilde{e} := \tilde{u} - \tilde{u}_h$, where $\tilde{u}_h$ denotes the regularized approximate solution. The a priori error estimates that we prove typically take the form

$$|||\tilde{e}(T)|||_\delta := \|\delta \nabla \tilde{e}(T)\|_\Omega + \|\tilde{e}(T)\|_\Omega \leq C(u_0, T) \exp\left(\tfrac{T}{\tau_F}\right) \beta^{\frac{1}{2}} \left(\frac{h}{\delta^2}\right)^{\frac{1}{2}} \tag{2}$$

where $\tilde{u}$ and $\tilde{u}_h$ are the filtered exact and computational solution respectively. The constant $C(u_0, T)$ in (2) depends only on the intial data, the mesh geometry and the final time and the coefficient β is an upper bound on the transport velocity. In some estimates length scales related to the $O(1)$ size of the domain have been omitted. The characteristic time τ_F depends on the velocity field in a nontrivial way and a key point in the below discussion is when τ_F can be expected to be $O(1)$ so that the exponential growth is moderate for moderate T. Note that the right hand side of (2) is independent of both the viscosity parameter and the Sobolev regularity of the exact solution. For previous work on error estimates for filtered solutions see [6], their estimates however are not robust in the Reynolds number.

The derivation of the estimate (2) uses:

– sharp energy stability estimates for the finite element method,

– L^∞-estimates for the finite element solution in the nonlinear cases,

– a priori stability estimates on a linearized dual problem with regularized data,

– Galerkin orthogonality and approximability.

To obtain precise control of all constants we must control the asymptotic growth of the residual and work with the exact dual adjoint, involving both the approximate and the exact solution in the nonlinear case. We will frequently use the notation $a \lesssim b$ defined by $a \leq Cb$ as well as $a \sim b$ meaning that $a \lesssim b$ and $b \lesssim a$ with C a constant independent of h, any essential physical parameters and of the exact solution. Some dependence on physical parameters may be included in the constants if it may be assumed not to change the magnitude of the constant.

2 THE BURGERS' EQUATION

Consider the simple model case of the Burgers' equation with periodic boundary conditions, on the space-time domain $Q := \Omega \times I$, with $\Omega := (0,1)$ and $I := (0,T)$ for some $T > 0$,

$$\begin{array}{rcl} \partial_t u + \frac{1}{2}\partial_x u^2 - \nu \partial_{xx} u & = & 0 \text{ in } Q \\ u(0,t) & = & u(1,t) \text{ for } t \in I \\ u(x,0) & = & u_0(x) \text{ for } x \in \Omega. \end{array} \tag{3}$$

First we discuss the $L^\infty(I; L^2(\Omega))$ stability of the Burgers' equation and conclude that the resulting estimate includes an exponential factor of the type $\exp(\|\partial_x u\|_{L^\infty} T)$ reflecting a possible instability in the L^2-norm. Then we introduce the finite element discretization and briefly discuss the stability properties of the method. Finally we consider filtering of the final solution and show that the perturbation equation corresponding to the filtered solution has improved stability properties and the error may therefore be upper bounded independently of both the regularity of the exact solution and the physical viscosity. As we shall see, although $\|(u-u_h)(\cdot,T)\|_\Omega$, where u_h denotes the finite element approximation of (3), does not appear to allow for error estimates with moderate constants, the L^2-error of the filtered error, $\|(\tilde{u} - \tilde{u}_h)(\cdot,T)\|_\Omega$ does. Indeed, for the Burgers' equation in the high Reynolds number regime we prove the error estimate

$$|||\tilde{u} - \tilde{u}_h|||_\delta \leq \tilde{C}(u_0,T)\exp(D_0 T)\left(\frac{h}{\delta^2}\right)^{\frac{1}{2}} \tag{4}$$

where $\tilde{u}$ and $\tilde{u}_h$ are the filtered exact and computational solution respectively and $D_0 \sim \sup_{x\in\Omega} |\partial_x u_0|$. We will also use the notation $U_0 \sim \sup_{x\in\Omega} |u_0(x)|$. For simplicity we assume $u_0 \in C^\infty(\Omega)$, this does not exclude the formation of sharp layers with gradients of order ν^{-1} at later times. For fixed filter width (4) results in a convergence rate of order $h^{\frac{1}{2}}$. If on the other hand the filter width is related to the mesh size $\delta \sim h^\alpha$ with $\alpha < \frac{1}{2}$ we get the convergence rate $h^{\frac{1-2\alpha}{2}}$. The parameter δ determines how strong the localization of the norm is. The choice $\delta = 1$ leads to a norm related to the H^{-1}-norm and the choice

$\delta = h$ leads to a norm similar to the L^2-norm. Clearly the estimates proposed here only makes sense for $0 \leq \alpha < \frac{1}{2}$. This indicates that no error bounds in a norm similar to the L^2 case can be obtained in this framework.

3 The Burgers' equation with viscous dissipation

The wellposedness of the equation (3) for $\nu \geq 0$ is well known it is also known that for $\nu > 0$ by parabolic regularization the solution is $C^\infty(\Omega)$. This high regularity however does not necessarily help us when approximating the solution, since we are interested in computations using a mesh-size that is much larger than the viscosity and still want the bounds to be Robust with respect to the Reynolds number.

3.1 L^2-stability of Burgers' equation

Consider a general perturbation $\eta(x)$ of the initial data of (3).

$$\begin{array}{rcl} \partial_t \hat{u} + \frac{1}{2}\partial_x \hat{u}^2 - \nu \partial_{xx} \hat{u} &=& 0 \text{ in } Q \\ \hat{u}(0,t) &=& \hat{u}(1,t) \text{ for } t \in I \\ \hat{u}(x,0) &=& u_0(x) + \eta(x) \text{ for } x \in \Omega. \end{array} \tag{5}$$

Taking the difference of (5) and (3) leads to the perturbation equation for $\hat{e} := \hat{u} - u$ with $a(u,\hat{u}) := \frac{1}{2}(u + \hat{u})$,

$$\begin{array}{rcl} \partial_t \hat{e} + \partial_x (a(u,\hat{u})\hat{e}) - \nu \partial_{xx} \hat{e} &=& 0 \text{ in } Q, \\ \hat{e}(0,t) &=& \hat{e}(1,t) \text{ for } t \in I \\ \hat{e}(x,0) &=& \eta(x) \text{ for } x \in \Omega. \end{array} \tag{6}$$

Multiplying equation (6) by $\hat{e}$ and integrating over Q leads to the energy equality

$$\frac{1}{2}\|\hat{e}(T)\|_\Omega^2 + \|\nu^{\frac{1}{2}} \partial_x \hat{e}\|_Q^2 = \frac{1}{2}\|\eta\|_\Omega^2 - \int_Q (\partial_x a(u,\hat{u}))\hat{e}^2.$$

We know that due to shock formation $-\partial_x a(u,\hat{u}) \sim \nu^{-1}$. Any attempt to obtain control of $\|\hat{e}(T)\|_\Omega^2$ in terms of the initial data will rely on Gronwall's lemma, leading to

$$\|\hat{e}(T)\|_\Omega^2 \leq C_a \|\eta\|_\Omega^2$$

with the exponential factor

$$C_a := \exp(\|\partial_x a(u,\hat{u})\|_{L^\infty(Q)} T) \sim \exp(T/\nu).$$

This estimate tells us that we have stability (and hence computability) only up to the formation of shocks. Using this type of argument in the analysis of the finite element method leads to error estimates useful only for solutions with moderate gradients.

3.2 Maximum principles for Burgers' equation

It is well known that the equation (3) satisfies a maximum principle on the form:

$$\sup_{(x,t)\in Q} |u(x,t)| \leq \sup_{x\in\Omega} |u_0(x)|. \tag{7}$$

For our purposes we also need some precise information on the derivative. Since the solution of (3) is smooth we may derive the equation in space to obtain the following equation for the space derivative $w := \partial_x u$:

$$\begin{aligned} \partial_t w + u\partial_x w - \nu\partial_{xx} w &= -w^2 \text{ in } Q \\ w(0,t) &= w(1,t) \text{ for } t \in I \\ w(x,0) &= \partial_x u_0(x) \text{ for } x \in \Omega. \end{aligned} \tag{8}$$

Assuming that w takes its maximum in some point $x \in I$ and noting that $\partial_x w(x) = 0$ and $\partial_{xx} w(x) < 0$ it follows that $\partial_t w < 0$ at the maximum and we deduce the bound:

$$\max_{(x,t)\in Q} \partial_x u \leq \max_{x\in\Omega} \partial_x u_0. \tag{9}$$

It follows by the smoothness of the initial data that the space derivative is bounded above for all times.

4 Artificial viscosity finite element method

Discretize the interval Ω with N elements and let the local mesh-size be defined by $h := 1/N$. We denote the computational nodes by $x_i := i\,h$, $i = 0, \dots, N$, defining the elements $\Omega_j := [x_j, x_{j+1}]$, $j = 0, \dots, N-1$. The finite element space is given by

$$V_h := \left\{ v_h \in H^1(\Omega) : v_h|_{\Omega_j} \in P_1(\Omega_j); u_h(0) = u_h(1) \right\}.$$

We define the standard L^2 inner product on $X \subset \Omega$ by $(v_h, w_h)_X := \int_X v_h w_h \,\mathrm{d}x$. The discrete form corresponding to mass-lumping reads $(v_h, w_h)_h := \sum_{i=0}^{N-1} v_h(x_i) w_h(x_i) h$. The associated norms are defined by $\|v\|_X := (v,v)_X^{\frac{1}{2}}$, for all $v \in L^2(X)$, if X coincides with Ω the subscript may be dropped, and $\|v_h\|_h := (v_h, v_h)_h^{\frac{1}{2}}$ for all $v_h \in V_h$. Note that, by norm equivalence on discrete spaces, for all $v_h \in V_h$ there holds $\|v_h\|_h \sim \|v_h\|$. Using the above notation the artificial viscosity finite element space semi-discretization of (3) writes, given $u_0 \in C^\infty(\Omega)$ find $u_h(t) \in V_h$ such that $(u_h(0), v_h)_\Omega = (u_0, v_h)_\Omega$ and

$$(\partial_t u_h, v_h)_h + \left(\partial_x \frac{u_h^2}{2}, v_h\right)_\Omega + (\hat{\nu}\partial_x u_h, \partial_x v_h)_\Omega = 0, \text{ for all } v_h \in V_h \text{ and } t > 0, \tag{10}$$

where we propose two different forms of $\hat{\nu}$:

1. linear artificial viscosity:
$$\hat{\nu} := \max(U_0 h/2, \nu); \tag{11}$$

2. nonlinear artificial viscosity:
   Let $0 \leq \epsilon$ and
$$\nu_0(u_h)|_{\Omega_i} := \frac{1}{2}\|u_h\|_{L^\infty(\Omega_i)} \max_{x \in \{x_i, x_{i+1}\}} \frac{|[\![\partial_x u_h]\!]|_x|}{2\{|\partial_x u_h|\}|_x + \epsilon}, \tag{12}$$
   where $[\![\partial_x u_h]\!]|_{x_i}$ denotes the jump of $\partial_x u_h$ over the node x_i and $\{|\partial_x u_h|\}|_{x_i}$ denotes the average of $|\partial_x u_h|$ over x_i. If $\epsilon = 0$ and $\{|\partial_x u_h|\}|_{x_i} = 0$ we replace the quotient $|[\![\partial_x u_h]\!]|_{x_i}|/\{|\partial_x u_h|\}|_{x_i}$ by zero.
   Further let
$$\xi(u_h)|_{\Omega_i} := \begin{cases} 1 & \text{if } \partial_x u_h|_{\Omega_i} > 0,\ \partial_x u_h|_{\Omega_i} > \partial_x u_h|_{\Omega_{i+1}} > 0 \\ & \text{and } \partial_x u_h|_{\Omega_i} \geq \partial_x u_h|_{\Omega_{i-1}} > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\nu_1(u_h)|_{\Omega_i} := \xi(u_h)|_{\Omega_i} \max\Big(\nu_0|_{\Omega_{i-1}} \frac{\partial_x u_h|_{\Omega_{i-1}}}{\partial_x u_h|_{I_i}}, \nu_0|_{\Omega_{i+1}} \frac{\partial_x u_h|_{\Omega_{i+1}}}{\partial_x u_h|_{I_i}}\Big). \tag{13}$$
   Finally define:
$$\hat{\nu}(u_h)|_{\Omega_i} := \max(\nu, h(\nu_0|_{\Omega_i} + \nu_1|_{\Omega_i})). \tag{14}$$

The rationale for the nonlinear viscosity is to add first order viscosity at local extrema of the solution u_h so that (7) holds also for the discrete solution and enough viscosity at positive extrema of $\partial_x u_h$, making (9) carry over to the discrete setting. Using the properties of the numerical viscosity we may prove the following discrete stability estimate.

The solution u_h of the formulation (10) with either the linear artificial viscosity given by (11) or the nonlinear one of (14) with $\epsilon = 0$, satisfies the upper bounds
$$\|u_h(T)\| + \|\hat{\nu}^{\frac{1}{2}} \partial_x u_h\|_Q \lesssim \|u_0\|, \quad \|\partial_t u_h\|_Q \lesssim (U_0 T^{\frac{1}{2}} h^{-\frac{1}{2}} + \nu^{\frac{1}{2}})\|\partial_x u_0\|. \tag{15}$$

4.1 Error estimates for the Burgers' equation

To derive error estimates in the norm $|||\cdot|||_\delta$ we introduce the linearized adjoint problem
$$\begin{aligned} -\partial_t \varphi + a(u, u_h)\partial_x \varphi - \nu \partial_{xx}\varphi &= 0 \text{ in } Q, \\ \varphi(0,t) &= \varphi(1,t) \text{ for } t \in I, \\ \varphi(x,T) &= \psi(x) \text{ for } x \in \Omega. \end{aligned} \tag{16}$$

The following stability estimate for (16) follows easily by standard energy methods since *both* the discrete and the continuous solutions satisfy maximum principles of the type (7) and (9),
$$\sup_{t \in (0,T)} \|\partial_x \varphi(\cdot,t)\|^2 + \nu\|\partial_{xx}\varphi\|_Q^2 \lesssim \exp(D_0 T)\|\partial_x \psi\|^2. \tag{17}$$

The rationale for the dual adjoint is the following derivation of a perturbation equation for the functional of the error $|(e(T),\psi)_\Omega|$, where $e(T) := u(T) - u_h(T)$.

$$\begin{aligned}|(e(T),\psi)_\Omega| &= |(e(T),\psi)_\Omega + \int_0^T (e, -\partial_t\varphi + a(u,u_h)\partial_x\varphi - \nu\partial_{xx}\varphi)_\Omega \,\mathrm{d}t| \\ &= |(e(0),\varphi(0))_\Omega - \int_0^T (\partial_t u_h + u_h\partial_x u_h, \varphi)_\Omega \,\mathrm{d}t - \int_0^T (\nu\partial_x u_h, \partial_x\varphi)_\Omega \,\mathrm{d}t|. \end{aligned} \quad (18)$$

This relation connects the error to the computational residual weighted with the solution to the adjoint problem and can lead both to a posteriori error estimates and to a priori error estimates, provided we have sufficient information on the stability properties of the numerical discretization methods and of the dual problem. Observing that

$$|||\tilde{e}(T)|||_\delta^2 = (\delta\partial_x\tilde{e}(T), \partial_x\tilde{e}(T))_\Omega + (\tilde{e}(T),\tilde{e}(T))_\Omega = (e(T),\tilde{e}(T))_\Omega \quad (19)$$

we deduce that the choice $\psi = \tilde{e}(T)$ in (16) leads to an error representation for the filtered error. Using this error representation, Galerkin orthogonality and the stability of the dual solution (17) we may prove the following a posteriori error estimate. The associated a priori error estimate is a direct consequence of the a posteriori error bound, the maximum principles satisfied by the discrete solution and the bounds of (15).

Theorem 1 *Let u be the solution of* (3)*, u_h be the solution of* (10)*. Then the following a posteriori and a priori bounds hold:*

$$\begin{aligned}|||\tilde{e}(T)|||_\delta \lesssim \exp(D_0T)\left(\frac{h}{\delta^2}\right)^{\frac{1}{2}} \Big(h^{\frac{1}{2}}\|(u-u_h)(0)\| + h^{\frac{1}{2}}\int_0^T \inf_{v_h\in V_h}\|v_h + u_h\partial_x u_h\| \; dt \\ + h^{\frac{3}{2}}\int_0^T \|\partial_x\partial_t u_h\| \; dt + \int_0^T \|\max(0,\hat{\nu}-\nu)^{\frac{1}{2}}\partial_x u_h\| \; dt + h\Big(\int_0^T \nu \|[\![\partial_x u_h]\!]\|_N^2 \; dt\Big)^{\frac{1}{2}}\Big), \end{aligned} \quad (20)$$

where $\|[\![\partial_x u_h]\!]\|_N := \left(\sum_{i=0}^{N-1}(\partial_x u_h(x_i)|_{\Omega_{i+1}} - \partial_x u_h(x_i)|_{\Omega_i})^2\right)^{\frac{1}{2}}$, *with Ω_N identified ith Ω_0 by periodicity.*

$$|||\tilde{e}|||_\delta \;\lesssim\; \exp(D_0T)\left(\frac{h}{\delta^2}\right)^{\frac{1}{2}} \Big(\Big(h^{\frac{1}{2}} + U_0^{\frac{1}{2}}\sqrt{T}\Big)\|u_0\| + (TU_0 + h^{\frac{1}{2}}\nu^{\frac{1}{2}})\|\partial_x u_0\|\Big). \quad (21)$$

5 EXTENSION TO FLOW IN HIGHER DIMENSION

In higher dimension the difficulty compared to the Burgers equation, is that the gradient tensor of the velocity can not be expected to have any sign, even when the flow is incompressible. If strong vortices or separation is present in the flow the diverging streamlines may cause exponential growth of perturbations with factor proportional to the maximum velocity gradient in the energy estimates. This reflects that two particles that initally are close may be separated very quickly by the flow, hence giving rise to sensitivity to perturbations. Below we will discuss how the idea of estimating filtered quantities can be used for the derivation of robust error estimates, first for passive transport with rough data and then for the two-dimensional Navier-Stokes equation. A key assumption in the below argument is a large eddy hypothesis, stating that the velocity field allows for an a priori decomposition where the main energy is carried by large eddies with moderate gradients and that remaining component can have arbitrary oscillation, but energy comparable to the diffusive/viscous dissipation, as made precise in this assumption.

Assumption 1 *(Large eddy scale separation)*
Let $\boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^2$. Given $\mu \in \mathbb{R}^+$, assume that there exists a decomposition of the velocity field,

$$\boldsymbol{\beta} = \overline{\boldsymbol{\beta}} + \boldsymbol{\beta}',$$

where, for all t, $\|\overline{\boldsymbol{\beta}}\|_{W^{1,\infty}(\Omega)} \sim 1$ and $\|\boldsymbol{\beta}'\|^2_{L^\infty(\Omega)} \sim \mu$.

Under this assumption we may define a global timescale for the flow relating to both the coarse scale spatial variation and the fine scale amplitude,

$$\tau_F := \min(\|\bar{\boldsymbol{\beta}}\|^{-1}_{W^{1,\infty}(\Omega)}, \mu/\|\boldsymbol{\beta}'\|^2_{L^\infty(\Omega)}) \sim 1. \tag{22}$$

Of course for any given $\boldsymbol{\beta}$ and viscosity μ one can find the optimal decomposition $\overline{\boldsymbol{\beta}} + \boldsymbol{\beta}'$ that maximizes τ_F, which gives a measure of the computability of that particular flow problem. Essentially we assume that the velocity vectorfield can be decomposed in a coarse scale, responsible for transport, that is slowly varying in space and a fine scale, responsible for mixing, that has small amplitude but may have very strong spatial variation. Expressed in Péclet numbers this means that the coarse scale Péclet number may be arbitrarily high, whereas the fine scale Péclet number must be of order one.

The Assumption 1 may now be used to derive a posteriori and a priori error estimate that are robust in the multidimensional case. We will briefly review the cases of passive transport and two dimensional Navier-Stokes' below.

5.1 Transient convection–diffusion equations

The problem that we will consider takes the following form. Let Ω be an open polygonal/polyhedral subset of $\mathbb{R}^d$, with boundary $\partial\Omega$, $u_0, f \in L^2(\Omega)$ and let $\boldsymbol{\beta} \in [C_0(I; W^{1,\infty}(\Omega))]^d$, $\mu \in \mathbb{R}^+$, then formally we may write, for $t > 0$ find $u \in H^1_0(\Omega)$ such that $u(x, 0) = u_0(x)$ in Ω and

$$\partial_t u + \boldsymbol{\beta} \cdot \nabla u - \mu \Delta u = f, \quad \text{in } \Omega. \tag{23}$$

For the boundary conditions let $u|_{\partial\Omega} = 0$ and assume that the velocity field satisfies non-penetration boundary conditions $\boldsymbol{\beta} \cdot n_{\partial\Omega}|_{\partial\Omega} = 0$. We also consider the associated dual problem, for $t > 0$ find $\varphi \in H^1_0(\Omega)$ such that

$$\begin{aligned} -\partial_t \varphi - \boldsymbol{\beta} \cdot \nabla \varphi - \mu \Delta \varphi &= 0 \text{ in } \Omega \\ \varphi &= 0 \text{ on } \partial\Omega \\ \varphi(\cdot, T) &= \psi(\cdot) \text{ in } \Omega. \end{aligned} \tag{24}$$

Using energy methods and the Assumption 1 we may prove the following stability estimate for the dual solution

$$\sup_{t \in I} |||\varphi(\cdot,t)|||_\delta + T^{-1} \|\delta^{1/2} \nabla \varphi\|_Q + T^{-1} \|\delta^{1/2} \partial_t \varphi\|_Q + \|(\delta\mu)^{1/2} \Delta \varphi\|_Q \lesssim C_{\tau_F,T} |||\psi|||_\delta, \tag{25}$$

with $C_{\tau_F,T} \sim e^{\left(\frac{T}{\tau_F}\right)}$, where τ_F is given by (22).

5.1.1 Finite element discretization

Let $\{\mathcal{T}_h\}_h$ be a family of nonoverlapping conforming, quasi uniform triangulations, $\mathcal{T}_h := \{K\}_h$ where the triangles K have diameter h_K and that is indexed by $h := \max h_K$. We let the set of interior faces $\{F\}_h$ of a triangulation $\mathcal{T}_h$ be denoted by $\mathcal{F}$.

We will consider a standard finite element space of piecewise affine, continuous functions $V_h := \{v_h \in H^1(\Omega) : v_h|_K \in P_1(K), \quad \forall K \in \mathcal{T}_h\}$, where $P_1(K)$ denotes the set of affine polynomials on K also let $V_h^0 := V_h \cap H^1_0(\Omega)$.

For $t > 0$ find $u_h \in V_h^0$ such that $u_h(x,0) = \pi_h u_0(x)$ and

$$(\partial_t u_h, v_h) + a(u_h, v_h) + s_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h^0, \tag{26}$$

where $a(\cdot,\cdot)$ is defined by:

$$a(u,v) := (\beta \cdot \nabla u, v) + (\mu \nabla u, \nabla v)$$

and

$$s_h(u_h, v_h) := \gamma \sum_{F \in \mathcal{F}} \left\langle h_F^2 \|\boldsymbol{\beta} \cdot n_F\|_{L^\infty(F)} [\![\nabla u_h \cdot n_F]\!], [\![\nabla v_h \cdot n_F]\!] \right\rangle_F. \tag{27}$$

The finite element method (26) satisfies the estimate

$$\sup_{t \in I} \|u_h(t)\|_\Omega + \|\mu \nabla u_h\|_Q + \left(\int_0^T s_h(u_h, u_h) \,\mathrm{d}t \right)^{\frac{1}{2}} \lesssim \int_0^T \|f\|_\Omega \,\mathrm{d}t + \|u_0\|_\Omega. \tag{28}$$

Theorem 2 *(A posteriori error estimate) Let $\tilde{e} := \tilde{u} - \tilde{u}_h$. Then there holds*

$$\begin{aligned} |||\tilde{e}|||_\delta \lesssim C_{\tau_F,T} \left(\frac{h}{\delta^2}\right)^{1/2} \Big(\int_I \inf_{v_h \in V_h} \|h^{1/2}(\boldsymbol{\beta} \cdot \nabla u_h - v_h)\|_\Omega \; dt \\ + \int_I \sum_{F \in \mathcal{F}} \left(\|\mu [\![\nabla u_h]\!]\|_F^2 \right)^{1/2} dt \\ + \int_I s_h(u_h,u_h)^{\frac{1}{2}} \, dt + h^{1/2} \int_I \|f - \pi_h f\|_\Omega \; dt + h^{1/2} \|u_0 - \pi_h u_0\|_\Omega \Big), \end{aligned} \tag{29}$$

where we recall that $C_{\tau_F,T} \sim e^{\left(\frac{T}{\tau_F}\right)}$.

Theorem 3 *(A priori error estimate) Assume that* $\frac{\|\overline{\boldsymbol{\beta}}\|_{L^\infty(Q))}h}{\mu} > 1$*, with* $\|\overline{\boldsymbol{\beta}}\|_{L^\infty(Q))} \sim 1$*, then there holds*

$$|||\tilde{e}|||_\delta \lesssim C_{\tau_F,T} \left(\frac{h}{\delta^2}\right)^{1/2} (h^{1/2} + T^{\frac{1}{2}} \left(\int_0^T \|f\|_\Omega \, dt + \|u_0\|_\Omega\right). \tag{30}$$

The right hand side of (30) is independent of μ and Sobolev norms of the solution. It only depends on the L^2-norm of data, showing that even for cases with rough source terms and initial data, such as those encountered in environmental flows, this estimate holds.

Note that the stability of the dual problem holds regardless of the numerical method used. The stabilization in the numerical method allows us to control the first residual in the a posteriori error estimate, by using the discrete stability estimate (28). If no stabilization is present there is no control of the streamline derivative, making it impossible to obtain uniformity in μ. If the domain is convex so that elliptic regularity can be used one may prove an optimal estimate valid also in the low Reynolds number regime

5.2 The Navier-Stokes' equations in two space dimensions

We will consider the Navier-Stokes' equations written on vorticity-velocity form. Let Ω be the unit square and assume that the boundary conditions are periodic in both cartesian directions. The equations then writes, $\omega(x,0) = \omega_0(x)$ and

$$\begin{aligned} \partial_t \omega + \nabla \cdot (u\omega) - \nu \Delta \omega = 0, \text{ in } Q, \\ -\Delta \Psi = \omega \text{ in } Q, \\ u = \text{rot } \Psi \text{ in } Q. \end{aligned} \tag{31}$$

Let $L_* := \{q \in L^2(\Omega);\ \int_\Omega q = 0\}$. The associated weak formulation takes the form for $t > 0$, find $(\omega, \Psi) \in H^1(\Omega) \times H^1(\Omega) \cap L_*(\Omega)$, with $\omega(x,0) = \omega_0(x)$ and such that for $t > 0$ and $\forall (v, \Phi) \in H^1(\Omega) \times H^1(\Omega) \cap L_*(\Omega)$,

$$\begin{aligned} (\partial_t \omega, v) + (\nabla \cdot (u\omega), v) + (\nu \nabla \omega, \nabla v) = 0, \\ (\nabla \Psi, \nabla \Phi) = (\omega, \Phi), \\ u = \text{rot } \Psi \text{ in } Q. \end{aligned} \tag{32}$$

6 Finite element discretization

Define V_h to be the standard space of piecewise affine, continuous periodic functions. Let $V_* := V_h \cap L_*$. We consider continuous finite elements with equal-order to discretize in space the vorticity ω and the stream function Ψ. The discrete velocity is given by

$u_h|_K := \text{rot } \Psi := \{\partial_y \Psi, -\partial_x \Psi\}$. Note that using this definition $\nabla \cdot u_h = 0$ in Ω, i.e. the discrete velocity is globally divergence free. For $t > 0$ find $\omega_h, \Psi_h \in V_h \times V_*$ such that

$$\begin{aligned}(\partial_t \omega_h, v_h)_M + (\nabla \cdot (u_h \omega_h), v_h) + (\nu \nabla \omega_h, \nabla v_h) + s(u_h; \omega_h, v_h) &= 0 \\ (\nabla \Psi_h, \nabla \Phi_h) - (\omega_h, \Phi_h) &= 0 \\ u_h - \text{ rot } \Psi_h &= 0, \quad \forall v_h, \Phi_h \in V_h \times V_*.\end{aligned} \tag{33}$$

Here $s(\cdot;\cdot,\cdot)$ denotes a stabilization operator that is linear in its last argument and $(\partial_t \omega_h, v_h)_M$ denotes the bilinear form defining the mass matrix, this operator either coincides with $(\cdot,\cdot)_\Omega$ or is defined as the scalar product $(\cdot,\cdot)_\Omega$ approximated using nodal quadrature, i.e. so called mass lumping. We will assume the stabilization term satisfies the bounds

$$\|h[\![u_h \cdot \nabla \omega_h]\!]\|_{\mathcal{F}} \lesssim s(u_h, \omega_h; \omega_h)^{\frac{1}{2}} \lesssim h^{\frac{1}{2}}(U_0 + \|u_h\|_{L^\infty(\Omega)})\|\nabla \omega_h\|,$$

$$s(u_h, \omega_h; v_h) \lesssim h^{\frac{1}{2}}(U_0^{\frac{1}{2}} + \|u_h\|_{L^\infty(\Omega)}^{\frac{1}{2}}) s(u_h, \omega_h; \omega_h)^{\frac{1}{2}} \|\nabla v_h\|.$$

This typically holds for (27) or for standard linear artificial viscosity with coefficient (11). The dual adjoint problem associated to the perturbation equation of (32) and (33) takes the form

$$\begin{aligned}-\partial_t \varphi_1 - u \cdot \nabla \varphi_1 - \varphi_2 - \nu \Delta \varphi_1 &= 0 \text{ in } Q, \\ -\Delta \varphi_2 - \nabla \omega_h \cdot \text{rot } \varphi_1 &= 0 \text{ in } Q, \\ \varphi_1(x, T) &= \psi(x) \text{ in } \Omega.\end{aligned} \tag{34}$$

A key result for the present analysis is the following stability estimate for the dual adjoint solution.

Proposition 1 *Assume that the exact velocity u satisfy the Assumption 1 with $\mu = \nu$. Then there holds for the solution (φ_1, φ_2) of (34),*

$$\sup_{t \in I} \|\nabla \varphi_1(\cdot, t)\| + \|\nu^{\frac{1}{2}} D^2 \varphi_1\|_Q \lesssim C_{\tau_F, T} \|\nabla \psi\| \tag{35}$$

$$\int_I \|\nabla \varphi_2(\cdot, t)\| \; dt \le C_{\tau_F, T} \int_I \|\omega_h\|_{L^\infty(\Omega)} \; dt \, \|\nabla \psi\|. \tag{36}$$

Using the dual problem with $\psi = \tilde{\omega} - \tilde{\omega}_h$ we may prove the following a posteriori estimate,

Theorem 4 *(A posteriori error estimates)*

$$\begin{aligned}|||\tilde{\omega} - \tilde{\omega}_h|||_\delta \lesssim e^{\frac{T}{\tau_F}} \left(\frac{h}{\delta^2}\right)^{\frac{1}{2}} \Big(&\|(\omega - \omega_h)(\cdot, 0)\| + \int_0^T \|h[\![u_h \cdot \nabla \omega_h]\!]\|_{\mathcal{F}} \; dt \\ &+ \int_0^T \|\nu^{\frac{1}{2}} [\![n_F \cdot \nabla \omega_h]\!]\|_{\mathcal{F}} \; dt + h^{\frac{1}{2}} \sup_{t \in I} \|\omega_h(\cdot, t)\| \int_0^T \|\omega_h(\cdot, t)\|_{L^\infty(\Omega)} \; dt \\ &+ \left(h^{\frac{3}{2}} \int_0^T \|\partial_t \nabla \omega_h\| \; dt\right)^* + \int_I s(u_h; \omega_h, \omega_h)^{\frac{1}{2}} \; dt \Big)\end{aligned} \tag{37}$$

where the term marked with a $*$ is omitted if the consistent mass matrix is used. For the velocities we have the estimate

$$\|(u-u_h)(\cdot,T)\| \leq \left(\|h^{\frac{1}{2}}[\![n_F\cdot\nabla\Psi_h(\cdot,T)]\!]\|_{\mathcal{F}} + |||(\tilde{\omega}-\tilde{\omega}_h)(\cdot,T)|||_1\right) \tag{38}$$

where $|||(\tilde{\omega}-\tilde{\omega}_h)(\cdot,T)|||_1$ may be a posteriori bounded by taking $\delta = 1$ in (37).

If we assume that $s_h(u_h,\omega_h,v_h)$ is strong enough so that $\|\omega_h\|_{L^\infty(Q)} \lesssim \|\omega_h(\cdot,0)\|_{L^\infty(\Omega)}$ then Theorem 4 together with the stability properties of the finite element method leads to the following a priori error estimates, that are independent of the Reynolds number and Sobolev norms of the exact solution,

$$|||(\tilde{\omega}-\tilde{\omega}_h)(T)|||_\delta \lesssim e^{\frac{T}{\tau_F}}\left(\frac{h}{\delta^2}\right)^{\frac{1}{2}} \text{ and } \|(u-u_h)(\cdot,T)\| \lesssim e^{\frac{T}{\tau_F}} h^{\frac{1}{2}}.$$

This can be achieved for instance using a linear artificial viscosity, similar to (11), or nonlinear diffusion of shock-capturing type (see [5] for precise definitions) on meshes for which the Laplacian produces an M-matrix, as detailed in [4].

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