P-ADAPTION FOR COMPRESSIBLE FLOWS

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Key words: Compressible flow, Discontinuous Galerkin method, p-adaption, Adaptive Modeling, Simulation

Abstract. We present a p-adaptive method which takes advantage of the ability of a discontinuity sensor used to quantify the difference between the actual solution and a projected reduced order solution in order to vary the polynomial resolution in an element.

1 INTRODUCTION

High-order methods have become increasingly more attractive in the field of aerodynamics due to their ability to increase the accuracy locally, their minimal numerical diffusion and dispersion properties, and the possibility to employ high-order meshes to better describe the geometry. The present work focuses on a spectral/hp element method using the Discontinuous Galerkin (DG) formulation [4] that is implemented using the open source library $Nektar^{++}$. The main advantages of the DG method range from its high accuracy to being highly flexible (allows for higher order meshes and h/p refinement) and its efficiency since it is easy to parallelise due to its block diagonal mass matrix structure. Although the DG formulation has numerous advantages, its main disadvantage is that it is computationally costly. A second limitation for the DG method is related to the treatment of flow discontinuities which, if approximated by a polynomial of high degree, leads to oscillations in the solution. As a result, an automatic polynomial adaptive procedure (p-adaption) is proposed in the present work.

The p-adaptive process can be applied to both inviscid and viscous flows and lead to a reduction in the computational cost of the simulation that could be significant, without loss of accuracy. A similar dynamic p-adaptive method is described in [1] and it is applied to the shallow water equations in [3]. The procedure described in these articles is based on a sensor that reconstructs the gradient of the solution and updates the polynomial degree by checking whether the magnitude of the sensor is higher or lower than a certain

threshold value. The method is applied at each time step and the adaption strategy is limited to use either $p = 1, 2$ or $p = 2, 3$ in each simulation.

The local support of the DG discretisation allows for the application of different polynomials and different number of quadrature points in different zones of the domain. Furthermore it is also possible to define the polynomial degree and the number of points of each element of the domain independently from each other. This property is intrinsic in the discontinuous features of the DG method. Since information is propagated between two elements only through their interface, the expansion within an element depends only upon its own values and the interface values of adjacent elements.

2   SENSOR-BASED P-ADAPTION STRATEGY

This study proposes an alternative strategy and the adaption procedure is applied after the current spatial distribution of polynomial degree, p, has converged to a steady solution. Moreover, the maximum degree is not imposed, but each element is free to assume any degree and the automatic p-adaptive strategy stops when a stable spatial p-distribution is reached. This method has been developed for modelling steady problems but it may be extended to time-dependent problems provided that an efficient method to vary the polynomial degree at each time step is implemented.

This procedure takes advantage of the ability of a discontinuity sensor used to quantify the difference between the actual solution (p) and the projected reduced one ($p - 1$) in order to vary the polynomial resolution in an element. The value of the sensor in an element is defined in the same way as described in [6]:

$$S_e = \frac{||\rho_e^p - \rho_e^{p-1}||_{L_2}}{||\rho_e^p||_{L_2}} \tag{1}$$

where ρ_e^p and ρ_e^{p-1} are the average solutions of degree p and $p - 1$ respectively on the same element. The polynomial degree is decreased when a discontinuity is present in order to avoid oscillations and increased when a high gradient is identified to improve the accuracy. This procedure allows the simulation to adapt to the flowfield, increasing the accuracy of the solution only where needed and, as a consequence, reducing the computational cost required for solving the problem. Furthermore, this sensor is used to locally add an extra diffusion term to enable shock capturing as described in [5, 6].

Initially, a converged linear solution is obtained after which the sensor in each element is calculated. Based on the determined sensor value and the pre-defined sensor thresholds, the degree of the polynomial approximation in each element is increased, reduced or maintained and a new converged solution is obtained. The sensor distribution is divided into four zones:

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$$p_e = \begin{cases} p_e - 1 & \text{if} \ \ s_e > s_{ds} \\ p_e + 1 & \text{if} \ \ s_{sm} < s_e < s_{ds} \\ p_e & \text{if} \ \ s_{fl} < s_e < s_{sm} \\ p_e - 1 & \text{if} \ \ s_e < s_{fl} \end{cases} \tag{2}$$

where s_{ds}, s_{sm} and s_{fl} are the threshold values to identify discontinuities, smooth and flat solutions respectively. This procedure is carried out iteratively.

In order to determine the solution at $p-1$ and threat the numerical fluxes at the edges of the element accordingly, the solution at polynomial order p, determined using a modified basis, has to be projected onto a hierarchical orthogonal expansion basis. Hence, using a more general formulation, the solution of a variable u is expressed as:

$$u^0 = u \tag{3}$$

where u and u^0 represent the general solution obtained using a modified basis ($\mathbf{B}$) and orthogonal ($\mathbf{B}^0$) respectively, hence:

$$\mathbf{B}^0 \hat{u}^0 = \mathbf{B}\hat{u} \rightarrow \hat{u}^0 = \left[\mathbf{B}^0\right]^{-1} \mathbf{B}\hat{u} \tag{4}$$

where $\hat{u}$ represents the vector of coefficients at polynomial p^+. Since a hierarchical basis is used, it is possible to lower the polynomial order to p^- where $p^+ > p^-$. Since the coefficients are not coupled in the hierarchical orthogonal basis and the information about the mean is contained only in the first coefficient, it is possible to apply a cut-off filter to the orthogonal coefficient vector. This cut-off filter sets all the coefficients that are higher than p^- equal to zero. The information contained in the high frequency components is removed without altering the mean value. The orthogonal coefficients represent the solution at p^- using an orthogonal basis, hence the following transformation has to be applied to obtain the modified filtered coefficients of the lower polynomial degree:

$$\hat{u}_f = \left[\mathbf{B}^{-1}\right] \mathbf{B}^0 \hat{u}_f^0 \tag{5}$$

The solution ρ_e^{p-1}, is obtained from ρ_e^p using this post processing step.

When dealing with different polynomial degrees, it is important to ensure an adequate treatment of the two following operations: the change of the polynomial degree of the solution in one element and the computation of the numerical flux on the interface. Hence, after the sensor is applied and the polynomial order of the element is changed, a similar filtering procedure is performed to compute the advective numerical fluxes on the interface of two elements with different expansions since the appropriate number of quadrature points has to be used. The number of quadrature points has to be equal to the number used by the highest polynomial degree of the two adjacent elements to avoid numerical

instabilities [2]. To ensure conservation and stability, the continuity of the total flux is required and therefore:

$$\int_{\Gamma_{f-}} \mathbf{F}_-^u d\Gamma_f = \int_{\Gamma_{f+}} \mathbf{F}_+^u d\Gamma_f \tag{6}$$

Where $\mathbf{F}_-^u$ and $\mathbf{F}_+^u$ represent the numerical flux on the edge between two elements with a lower and a higher polynomial order respectively. If the order or the quadrature points is different, the coefficients are copied directly on the higher resolved side, but fewer coefficients have to be set on the other side. The interface flux is then projected on the space of orthogonal polynomials and then filtered to delete the high-order frequencies. Once the degree of the orthogonal expansion is decreased to the lower degree, a reverse projection is carried out and modified coefficients are found.

3 RESULTS

The performance of the p-adaptive method is illustrated for the solution of the transonic flow ($M = 0.8$) over a NACA0012 aerosol under an angle of attack of $\alpha = 1.25°$. Two shocks are generated on the aerofoil: a strong shock on the top, at about $x = 0.6$ and a weaker shock on the bottom of the aerofoil at $x = 0.3$. The reference C_p distribution used for comparison is taken from [8], in which the numerical solution is obtained with a finite volume method, the aerofoil wall is discretised by 320 cells and the farfield boundary is placed at 25 chords. Figure 1 depicts the density and Mach distribution around the aerofoil, the final spatial p distribution and the sensor distribution. Even though the grid is very coarse, the shock is well resolved, it is captured in only one cell and it does not create oscillations in the neighbour cells. The discontinuity sensor is active only at the shock waves and the rest of the flow field is diffusion free. The p-adaptive procedure increases the polynomial degree of the discretisation close to the aerofoil and maintains $p = 3$ on the shock in order to avoid oscillations of the solution. Since the mesh is very coarse, most of the error in the C_p calculation is introduced at the shock position on the top of the aerofoil. Since the p-adaptive procedure does not increase the polynomial degree in the elements where the shock is present.

The efficiency of the p-adaptive procedure comes from the reduction in the number of operations required to solve the equations, but also the initial condition of each $1 \le p \le p_{max}$ simulation is a converged solution obtained with a lower degree ($1 \le p \le p_{max} - 1$). However, the smaller CFL time restriction associated with p_{max} has to be imposed over all the domain, thus reducing the time step also in the regions with lower polynomial order. A possible improvement could be the application of a domain decomposition technique or variable timestepping to deal with different values of Δt through the domain.

Current work is performed on the extension of the illustrated p-adaption technique to time-dependent 3D problems in $Nektar^{++}$ provided that an efficient method is developed

to vary the polynomial degree at each time step. Furthermore, ongoing investigation is performed on the topic of shock capturing in 3D compressible flow and results of both topics will be discussed during the ADMOS conference.
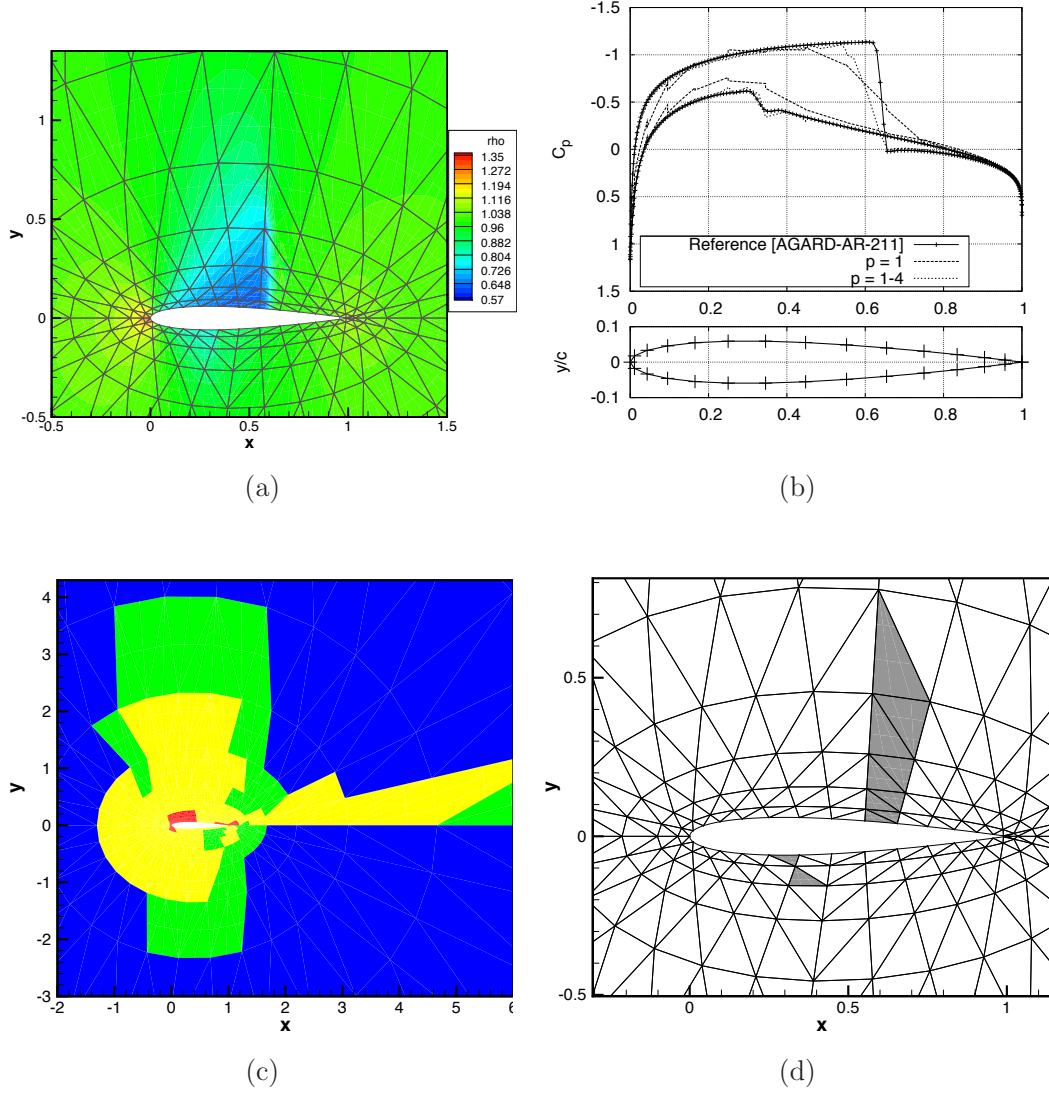

(a)

(b)

(c)

(d)

Figure 1: Solution of the inviscid transonic flow past a NACA0012 aerosol. (a) Density distribution; (b) Mach Distribution; (c) Polynomial degree distribution: blue: p = 1; green: p = ; yellow: p = 3; red: p = 4; (d) Sensor distribution.

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