VI International Conference on Adaptive Modeling and Simulation
ADMOS 2013
J. P. Moitinho de Almeida, P. Díez, C. Tiago and N. Parés (Eds)

ACCURATE MODELLING OF STRAIN DISCONTINUITIES IN BEAMS USING AN XFEM APPROACH

S. RAMAN*, B.C.N.MERCATORIS† AND A.DERAEMAEKER*

*Department of Building, Architecture and Town Planning (BATir)
Université Libre de Bruxelles
Avenue FD Roosevelt 50, CP 194/2, 1050 Brussels, Belgium
e-mail: sraman@ulb.ac.be,aderaema@ulb.ac.be, Website:batir.ulb.ac.be

†Department of Environmental Sciences and Technologies
Faculty of Gembloux Agro-Bio Tech
University of Liège
Passage des Dèportès 2, 5030 Gembloux, Belgium
e-mail: benoit.mercatoris@ulg.ac.be

Key words: XFEM,Timoshenko beams, Euler-Bernoulli beams, Assumed natural strain method, Shear locking

Abstract. The aim of this paper is to study the possibility of using extended finite element methods to model piezoelectric transducers attached to beam structures without the need for a conforming mesh. The main focus of this study is to propose enrichment functions to represent accurately the strain discontinuities in Euler-Bernoulli and Timoshenko beams. Further, we evaluate the performance of the enrichment functions on simple static cases with a special emphasis on the shear locking in the Timoshenko beam.

1 INTRODUCTION

Thin piezoelectric transducers are widely used in applications such as active vibration control, wave generation in materials and structural health monitoring. The finite element modelling of piezoelectric transducers is well established; an overview of the existing models can be found in [1]. Current practice for the modelling of structures equipped with flat piezoelectric transducers requires the development of specific beam or plate elements which are usually not available in commercial codes. The most important criteria when using the finite element method to model piezoelectric transducers attached to host structures is that the mesh must exactly match the boundary between the piezoelectric transducers and the host structure. This requirement of conforming meshes leads to extensive remeshing of the structure when optimal transducers configurations are investigated.

The need for conforming meshes arises due to the following reasons: the occurrence of a strain jump across the interface between the piezoelectric transducer and the host structure due to the additionnal stiffness of the piezoelectric transducer and the distributed efforts acting on the edges of the patch when used as an actuator, the continuity of the displacement field across the interface, and the presence of an electric field only in the piezoelectric material. To overcome meshing difficulties and capture local phenomenon, the extended finite element method (XFEM) for weak discontinuities was proposed for two-dimensional problems [2]. In this paper, we will make use of XFEM to develop enriched Euler-Bernoulli and Timoshenko beam elements that can capture jumps in strains across the interface between two materials using a non-conforming mesh. We identify the location of the interface using an implicit level-set method. This paper is organized as follows: Section 2 gives a brief overview about the behaviour of piezoelectric transducers under actuation and their impact on the host structures. In Section 3, we develop the enriched Euler-Bernoulli beam element with special emphasis on finding the right enrichment function. In section 4, we develop the enriched Timoshenko beam finite elements. In section 5, the shear locking problem of the enriched Timoshenko beams are explained and an assumed natural strain method to avoid shear locking are proposed and tested for an enriched Timoshenko beam element.

2 PIEZOELECTRIC ACTUATORS ATTACHED TO HOST STRUCTURES

Piezoelectric transducers operate in two modes: sensors and actuators. They are used as sensors when generation of a surface charge happens as a result of mechanically straining the piezoelectric material. For instance, this effect is usually used in force and acceleration sensors. They also function as actuators when the geometry of the piezoelectric material changes due to an applied electric field. Actuating the piezoelectric transducer produces equivalent forces on the host structures as described in [3]. A cantilever beam with attached piezoelectric transducers is considered in this paper as shown in Figure 1.

Actuating the piezoelectric transducers produces bending moments and point-forces in the host structure at the boundaries of the piezoelectric actuator as shown in Figure 2. These stresses result in jumps in the membrane deformation and curvature of the beam. In this study, we are interested in capturing these jumps using a non-conforming mesh. Specifically we will be tackling the jump in the curvature of beams. This case is similar to two-material beams where deformation jumps occur at the material interfaces. For the sake of simplicity, a two-material beam considered here in order to develop the enriched beam finite elements. These enriched finite elements could then be adapted to a multi-layer coupled electro-mechanical beam model.

3 ENRICHED EULER-BERNOULLI BEAM FINITE ELEMENT

Assuming Euler-Bernoulli theory, the transverse deflection w of the beam is governed by the fourth order differential equation given by

$$\frac{d^2}{dx^2}(EI\frac{d^2w}{dx^2}) = q(x) \tag{1}$$

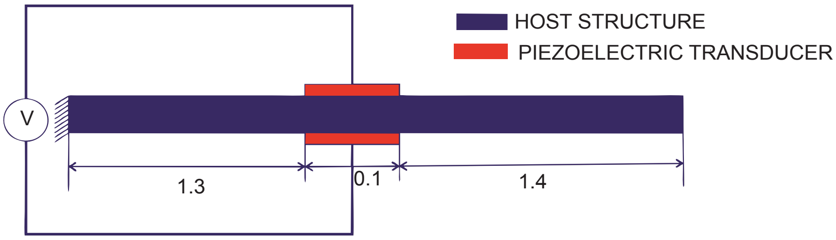

Figure 1: Cantilever beam with piezoelectric transducers

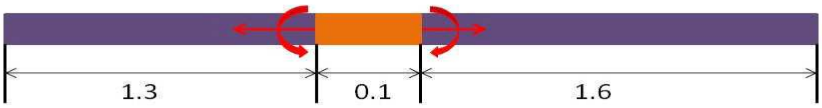

Figure 2: Equivalent loads on a cantilever beam with piezoelectric transducers

where E is the Young's Modulus of the beam, I is the area moment of inertia about the transverse axis of the beam and q is the distributed transverse load. The weak form of this equation is given for an element by

$$\int_{x_e}^{x_{e+1}} \left(EI \frac{d^2v}{dx^2} \frac{d^2w}{dx^2} - vq \right) dx + \left[v \frac{d}{dx} \left(EI \frac{d^2w}{dx^2} \right) - \frac{dv}{dx} EI \frac{d^2w}{dx^2} \right]_{x_e}^{x_{e+1}} = 0 \tag{2}$$

The essential boundary conditions involve the specification of the deflection w and the slope $\frac{dw}{dx}$. The natural boundary conditions involve the specification of the bending moment $EI\frac{d^2w}{dx^2}$ and the shear force $\frac{d}{dx}(EI\frac{d^2w}{dx^2})$ at the boundaries. The curvature of the beam is given by $\frac{d^2w}{dx^2}$ and is considered as the generalized strain measure of the beam. Hermite shape functions are used to approximate the deflection at any point of the beam using finite elements, which reads

$$u^{FEM} = \Sigma_{i=1}^{2}(H_i w_i + R_i \theta_i) \tag{3}$$

where u^{FEM} represents the deflection of the beam, H_i and R_i are the Hermite cubic shape functions and w_i and θ_i are the nodal deflections and nodal rotations. Considering the problem defined in Figure 2, classical finite element modelling requires conforming meshes in order to capture properly the material interfaces. An extended finite element method (XFEM) allows the use of non-conforming meshes. In a XFEM approach, the displacement field is enriched using the partition of unity technique [4] and is given by

$$u^{XFEM} = \Sigma_{i=1}^{2}(H_i w_i + R_i \theta_i) + \Sigma_{j=1}^{n} N_j \psi_j a_j \tag{4}$$

where N_j are the partition of unity shape functions, n is the total number of functions forming the partition of unity, ψ_j are the enrichment functions and a_j are the additional degrees of freedom related to the enrichment.

3.1 Enriched Nodes

The location of an interface is found using the implicit level-set method. The level-set is a measure of the signed distance between a node and the considered discontinuity. The elements where the discontinuity is present are found using level-sets as described in [5]. The kinematics of all the nodes belonging to this element needs to be enriched.

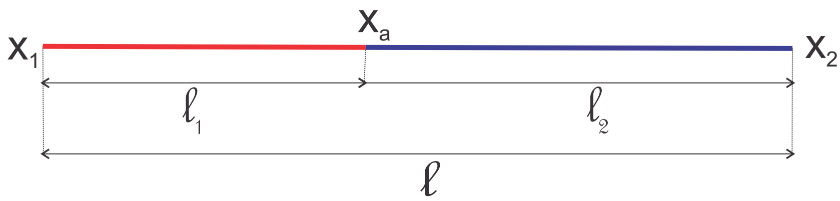

Figure 3: Enriched beam element

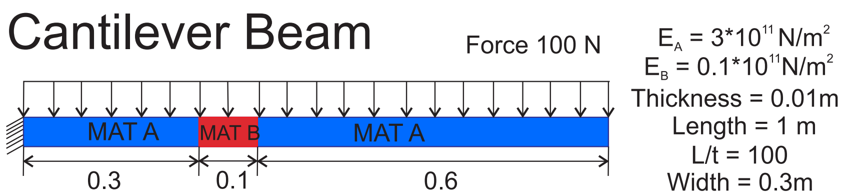

Figure 4: Two-material Cantilever beam with uniform load

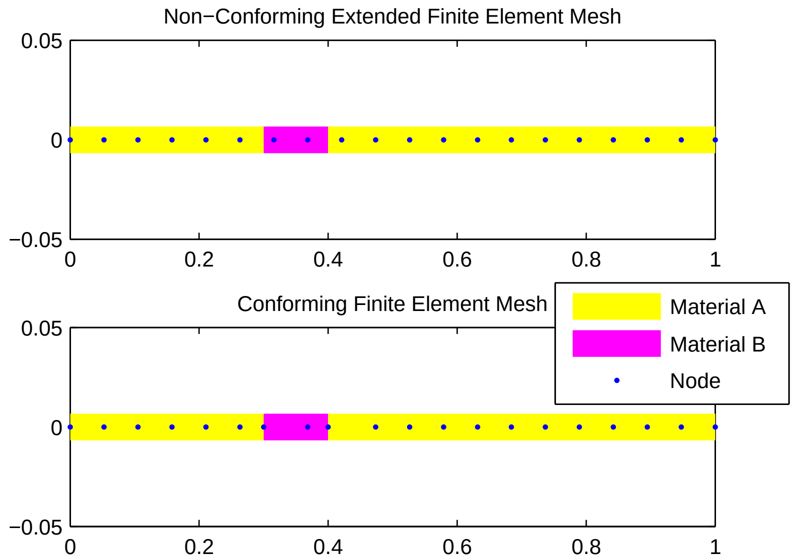

Figure 5: Non-conforming and conforming meshes for the considered problem

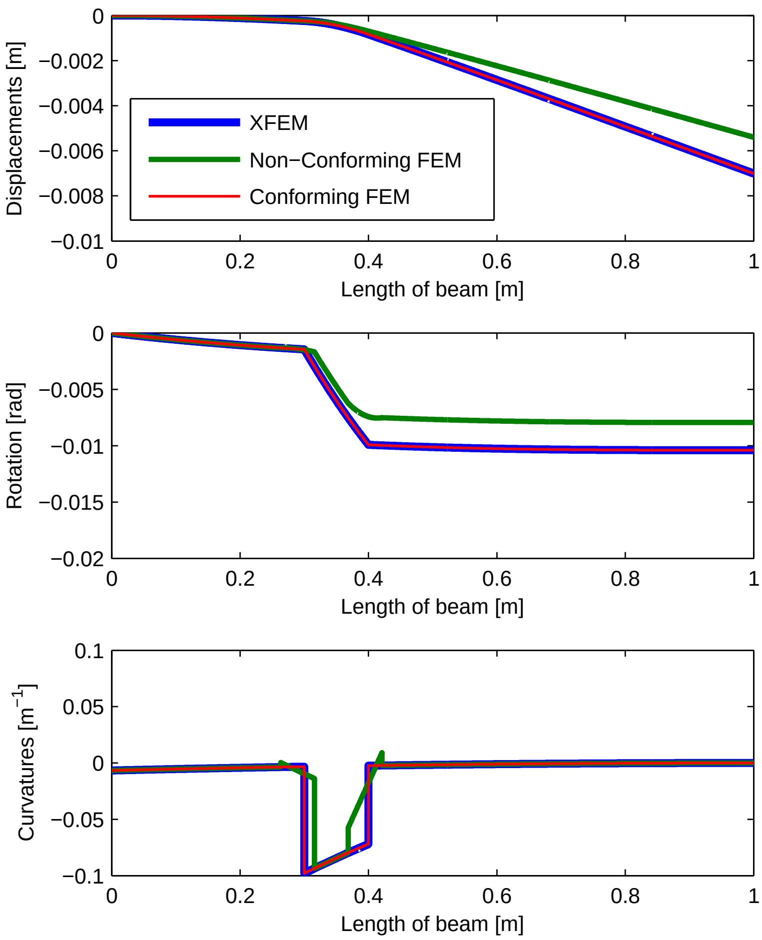

Figure 6: Comparison of Displacements, Rotations and Curvature of two-material Euler-Bernoulli beam

3.2 Enrichment functions for Euler-Bernoulli beams

In the field of XFEM, two-material problems are classified as weak discontinuity problems because the discontinuity occurs in the derivatives of the primary variables which remain continuous. The modelling of weak discontinuities using XFEM is explained in [2] and [5] for two-dimensional problems. In this case, the primary variable is discretized using linear shape functions and the enrichment functions are therefore not suitable for the case of Euler-Bernoulli beams which require cubic shape functions. Since two independent jumps, a curvature jump and a transverse shear jump, are needed to incorporate properly a discontinuity in an Euler-Bernoulli beam, two enrichment functions have to be defined. Their properties are listed as follows

- The enrichment functions have to be cubic and piece-wise continuous
- They have to vanish at the boundary of the element to avoid problems with blending elements as documented in [6]
- The first derivative of enrichment functions have to be continuous at the interface
- The first derivative of the enrichment functions should also get to zero at the boundary of the element
- The second and third derivatives of the enrichment functions should be discontinuous at the location of the interface

Considering an element depicted in Figure 3 whose extremities are located at X_1 and X_2 with a material discontinuity located at X_a, the enrichment functions are derived by means of the conditions set forth above. From the admissible space, the following set of enrichment functions are found

$$\psi_1 = \begin{cases} 3s_1^2 - 2s_1^3 & \text{if} \quad x < X_a \\ 1 - 3s_2^2 + 2s_2^3 & \text{if} \quad x > X_a \end{cases} \tag{5}$$

$$\psi_2 = \begin{cases} \ell_1 s_1^2(s_1 - 1) & \text{if} \quad x < X_a \\ \ell_2 s_2(s_2 - 1)^2 & \text{if} \quad x > X_a \end{cases}$$

where

$$s_1 = \frac{x - X_1}{\ell_1} \quad \text{and} \quad s_2 = \frac{x - X_a}{\ell_2} \tag{6}$$

3.3 Partition of unity for Euler-Bernoulli beams

Partition of unity is formed by a set of shape functions which add up to one. In case of Euler-Bernoulli beams, the H_i functions sum up to one and form a partition of unity. With the partition of unity established, we can now write the discretized XFEM expression for the beam element shown in Figure 3 as

$$u^{XFEM} = H_1 w_1 + +R_1\theta_1 + H_2 w_2 + R_2\theta_2 + H_1\psi_1 a_1 + H_2\psi_1 a_2 + H_1\psi_2 a_3 + H_2\psi_2 a_4 \tag{7}$$

with degrees of freedom w_1,θ_1,a_1 and a_3 at node 1 and the remaining at node 2.

3.4 Implementation

Considering the beam problem defined in Figure 4 and the conforming and non-conforming meshes depicted in Figure 5, the XFEM solution and the FEM solution on the same non-conforming mesh are plotted in Figure 6. These solutions are compared with the conforming mesh solution also given in Figure 6. It can be observed that the jump in curvature is properly captured by the XFEM using a non-conforming mesh. For the FEM solution using a non-conforming mesh,the change in material properties is carefully accounted for during numerical integration by applying the material property of the material where the integration point is located.

4 ENRICHED TIMOSHENKO BEAM FINITE ELEMENT

In this second part, the proposed approach is extended in order to incorporate a weak discontinuity within a Timoshenko beam theory. The strain energy considering both bending and shear contributions is given as

$$U = \frac{1}{2}\int_V \sigma_x \epsilon_x dV + \frac{1}{2}\int_V \tau_{xy}\gamma_{xy} dV \tag{8}$$

where the normal stresses are given by the Hooke's law

$$\sigma_x = E\epsilon_x \tag{9}$$

and the transverse shear stress is given by

$$\tau_{xy} = kG\gamma_{xy} \tag{10}$$

where k is the shear correction factor that varies according to the cross-section of the beam and G is the shear modulus given by

$$G = \frac{E}{2(1+\upsilon)} \tag{11}$$

where υ is the Poisson's ratio for the beam material. Considering a Timoshenko beam theory, the strain energy becomes

$$U = \frac{1}{2}\int_0^l EI(\frac{\partial \theta}{\partial x})^2 dx + \frac{1}{2}\int_0^l kAG(-\frac{\partial w}{\partial x} + \theta)^2 dx \tag{12}$$

In contrast with the Euler-Bernoulli beam element, independent linear interpolations are used here for the rotation and the deflection. For the case of a classical finite element, this reads

$$\begin{aligned} w^{FEM} &= N_1 w_1 + N_2 w_2 \\ \theta^{FEM} &= N_1\theta_1 + N_2\theta_2 \end{aligned} \tag{13}$$

where $N_1 = 1 - x/l$ and $N_2 = x/l$ where l is the length of the element. As with an Euler-Bernoulli beam, the Timoshenko beams can be enriched by introducing the partition of unity based enrichment functions as follows

$$\begin{aligned} w^{XFEM} &= N_1 w_1 + N_2 w_2 + N_1\psi_1 a_1 + N_2\psi_1 a_2 \\ \theta^{XFEM} &= N_1\theta_1 + N_2\theta_2 + N_1\psi_2 a_3 + N_2\psi_2 a_4 \end{aligned} \tag{14}$$

Since linear interpolations are used, the enrichment functions ψ_1 and ψ_2 can be ramp functions as described in [5]. The partition of unity is formed by the shape functions N_1 and N_2. As described in the previous section, the location of the discontinuity is found using the level-set and the elements containing a discontinuity are enriched.

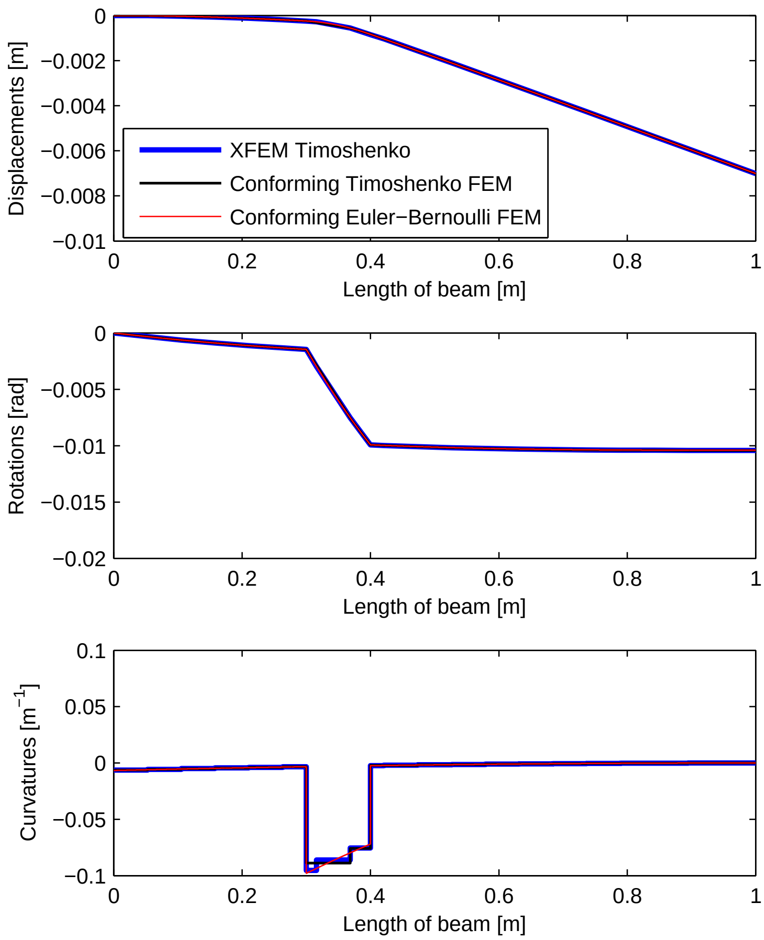

Figure 7: Comparison of the novel XFEM Timoshenko beam theory with respect of the clasical conforming beam theories for the two-material beam problem. The deflection, the rotation and the curvature are compared.

4.1 Locking in Timoshenko beams

Without using any specific treatment, the element beam described in the previous section suffers from shear locking. This occurs due to inconsistent interpolation for w and θ. In order to avoid the occurence of shear locking, many techniques were proposed such as the assumed natural strain method, the reduced integration method and the consistent interpolated element method. A detailed description of these methods can be found in [7]. In this paper, an assumed natural strain method is used to avoid any shear locking in the enriched Timoshenko beam element.

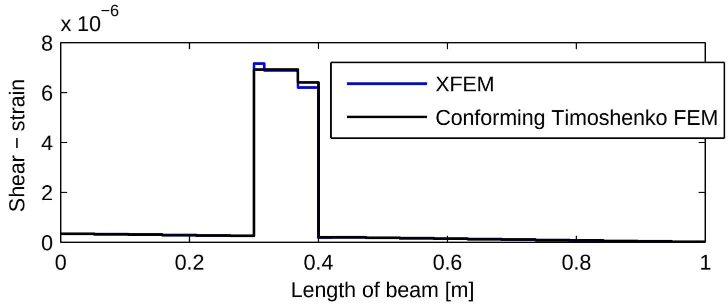

Figure 8: Comparison of shear-strains of two-material Timoshenko beam

5 SHEAR LOCKING TREATMENT IN AN ENRICHED TIMOSHENKO BEAM ELEMENT

The strain energy in Equation (12) is made of two parts namely the bending energy and the shear energy. The stiffness matrix can therefore be split into two parts: the bending part and the shear part which reads

$$K = \int_{x_1}^{x_2} [B^b]^T EI[B^b]dx + \int_{x_1}^{x_2} [B^s]^T kGA[B^s]dx \tag{15}$$

where B^b and B^s are the operators linking respectively the curvature and the transverse shear to the degrees of freedom of the beam. Using Equation (14), these operators are given for an XFEM beam element as

$$B^b = \begin{bmatrix} 0 & \frac{dN_1}{dx} & 0 & \frac{dN_2}{dx} & 0 & 0 & \frac{d(N_1\psi)}{dx} & \frac{d(N_2\psi)}{dx} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$$B^s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{dN_1}{dx} & N_1 & -\frac{dN_2}{dx} & N_2 & -\frac{d(N_1\psi)}{dx} & -\frac{d(N_2\psi)}{dx} & N_1\psi & N_2\psi \end{bmatrix}$$

The B^s matrix in the above equation will lead to a shear locking problem due to the presence of shape functions and their derivatives together. Based on the definition of the transverse shear strain $\chi = \theta - \frac{dw}{dx}$, an assumed natural strain method is used to treat properly the shear locking. The transverse shear strain is assumed piece-wise constant on each side of the interface. This is motivated by the fact that a conforming mesh with regular beam element would lead to a piece-wise constant transverse shear field with a discontinuity at the material interface. A classical collocation method is used to determine the assumed strain as follows

$$\begin{aligned} &\textstyle\int_{x_1}^{x_a} \overleftrightarrow{\chi_1} - \chi = 0 \\ &\textstyle\int_{x_a}^{x_2} \overleftrightarrow{\chi_2} - \chi = 0 \end{aligned} \tag{17}$$

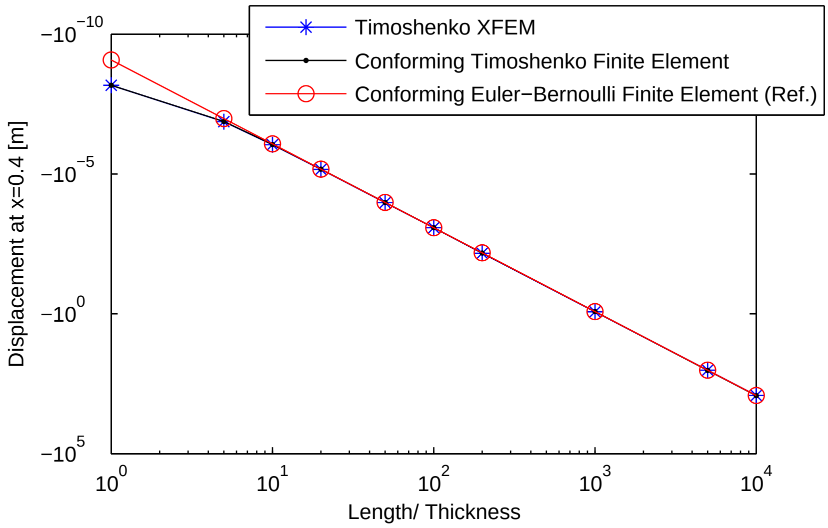

Figure 9: Behaviour of Elements as t→0

Expanding the above equations with the assumption that χ_1 and χ_2 are constant leads to

$$\overleftrightarrow{\chi_1} = \frac{1}{\ell_1}\int_{x_1}^{x_a} N_1\theta_1 + N_2\theta_2 + N_1\psi b_1 + N_2\psi b_2 \tag{18}$$
$$-(\frac{dN_1}{dx}w_1 + \frac{dN_2}{dx}w_2 + \frac{d(N_1\psi)}{dx}a_1 + \frac{d(N_2\psi)}{dx}a_2)$$

$$\overleftrightarrow{\chi_2} = \frac{1}{\ell_2}\int_{x_a}^{x_2} N_1\theta_1 + N_2\theta_2 + N_1\psi b_1 + N_2\psi b_2 \tag{19}$$
$$-(\frac{dN_1}{dx}w_1 + \frac{dN_2}{dx}w_2 + \frac{d(N_1\psi)}{dx}a_1 + \frac{d(N_2\psi)}{dx}a_2)$$

Using Equations (19) and (2), the B^s operator can be split into two contributions related to each part of enriched beam element as follows

$$B_1^s = \tfrac{1}{\ell_1}\int_{x_1}^{x_a}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{dN_1}{dx} & N_1 & -\dfrac{dN_2}{dx} & N_2 & -\dfrac{d(N_1\psi)}{dx} & -\dfrac{d(N_2\psi)}{dx} & N_1\psi & N_2\psi \end{bmatrix} \tag{20}$$

and

$$B_2^s = \tfrac{1}{\ell_2}\int_{x_a}^{x_2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{dN_1}{dx} & N_1 & \dfrac{dN_2}{dx} & N_2 & -\dfrac{d(N_1\psi)}{dx} & -\dfrac{d(N_2\psi)}{dx} & N_1\psi & N_2\psi \end{bmatrix} \tag{21}$$

5.1 Implementation

The considered problem is depicted in Figure 4. The Poisson's ratio is assumed to be 0.3 for both the materials. The deflection, the rotation and the curvature are compared in Figure 7 for the proposed XFEM Timoshenko formulation using the non-conforming mesh shown in Figure 5, with respect to a Timoshenko formulation and an Euler-Bernoulli formulation both using a refined conforming mesh. Since the beam is very thin, it is shown that the results of both Euler-Bernoulli and Timoshenko formulations are coherent. Also from Figure 8, it can be observed that the jumps in the transverse shear strain field is also captured accurately using XFEM. As shown in Figure 9, the solution of the enriched Timoshenko formulation and the solutions of conforming Timoshenko and Euler-Bernoulli approches are in good agreement when the beam thickness tends to zero. This effectively proves that any shear locking does not occur when using the extended Timoshenko beam finite element. The difference between the Euler-Bernoulli approach and the Timoshenko approach can also be observed in Figure 9, where for smaller values of length over thickness ratio, the solutions of the two approaches are different because the shear effects which are more prevalent at these ratios are only considered when modelling using the Timoshenko theory. Also the contribution from strain-energy to the total energy of the beam reduces as the length to thickness ratio increases. The enriched element behaves no different in this regard.

6 CONCLUSION AND PERSPECTIVES

In this study, we have detailed the enrichment functions for Euler-Bernoulli beam. Simple static case was tested with the newly proposed enrichment function and found to be satisfactory. A shear-locking free enriched Timoshenko beam finite element was also developed and tested for the same static case. An assumed natural strain method was used in order to avoid shear locking in Timoshenko beams. The current work will form the basis for the development of plate elements using an enriched finite element approach.

Acknowledgments

The first author would like to acknowledge the funding from FP7-Marie Curie Initial Training Network on Advanced Techniques in Computational Mechanics (ATCoMe).

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