

# EFFECTIVE APPLICATION OF THE EQUILIBRATED RESIDUAL METHOD IN ERROR ESTIMATION OF THE 3D-BASED *hp*-APPROXIMATED MODELS OF COMPLEX STRUCTURES

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**Abstract.** This paper recalls our previous research on adaptive modeling and analysis of structures of complex mechanical description. Such complex description results from the application of at least two different models for the structure mechanical characterization. The geometry of the structures can be either complex, i.e. composed of solid, shell and transition parts, or simple – with one geometrical part employed. The numerical models applied in such structures’ adaptive modelling and analysis is based on 3D-based hierarchical modelling and hierarchical *hp*-approximations. The corresponding control of the model and discretization adaptivities takes advantage of the a posteriori error estimation, which is based on the equilibrated residual method (ERM). The method is generalized for the special needs of 3D-based hierarchical models, and is applied to the assessment of the total and approximation error estimators/indicators. The modelling error indicators are obtained as the differences between their total and approximation counterparts. The necessary modifications of the original ERM are the first subject of this paper. These modifications concern both theoretical and implementation aspects. The second subject presented in this paper concerns the parametric studies of the global estimators or indicators of the total, modeling, and approximation error components. Various factors affecting effectivity of the estimation are taken into account.

## 1 INTRODUCTION

The paper completes our hitherto theoretical research efforts concerning the a posteriori error estimation in the case of complex structures. Such structures include at least two different mechanical models (theories) applied in mechanical characteristics of the structure. Within such structures we apply 3D-based approach utilizing only three-dimensional degrees of freedom. The 3D-based models for complex structures were presented in [1, 2]. The considered estimation method for such models is based on the equilibrated residual approach (ERM) [3] and is applied to the assessment of the global, total and approximation errors. The global modeling error is obtained as a difference of the former two errors. The results from the a posteriori error estimation are assigned for adaptivity control of hierarchical modeling and adaptive analysis of solid mechanics problems. These problems may correspond to either simple or complex mechanical description.

The global modeling error estimate and the element contributions to it allow for the adaptive hierarchical modeling within first order shell, hierarchical shell and the corresponding transition (either shell-to-shell or solid-to-shell) domains of the complex structures. Both, the change of the mechanical model or  $q$ -adaptivity are possible – with  $q$  denoting the transverse order of approximation within the hierarchical shell models. In the recalled approach also adaptive 2D, 3D or mixed (2D/3D)  $hp$ -approximations are possible in the shell, solid and transition domains of the complex structures, respectively, with  $h$  standing for the averaged element dimension and  $p$  denoting the longitudinal or three-dimensional order of approximation. The element contributions to the estimated global approximation error serve these two types ( $h$  and  $p$ ) of adaptivity.

Taking the above context into account, it is very important to have the estimation method which can satisfy the specific needs of the complex structures 3D-based modeling and analysis and delivers sufficiently accurate estimated values of the global errors, and acceptable element contributions to them as well. In order to satisfy the mentioned needs we adopt the existing algorithms of the equilibrated residual method. So far the method was applied to either the approximation error estimation within three-dimensional elasticity [4] or the total error estimation of the conventional hierarchical shell models [5, 6]. Also the approximation error estimation for the 3D-based first-order shell models is available [1, 7, 8]. Here we extent the application of the residual equilibrated method onto the estimation of the total error of the 3D-based first-order shell model, as well as the estimation of the total and approximation errors of the 3D-based hierarchical shell and transition models, both skipped in our previous works. In particular we show how to apply this method to the 3D-based (constrained) shell model of the first order and the corresponding constrained transition models as well. Such an application needs different equilibration procedure than the three-dimensional equilibration applied to the 3D-elasticity or hierarchical shells, where the equilibration is performed in the global directions. The adopted approach requires introduction of the local nodal coordinate systems, and different treatment of the constrained and unconstrained directions. Note

that some key aspects of this approach were presented in [9]. The present paper provides readers with some additional, theoretical and algorithmic, remarks or hints which were not published in the cited paper.

In order to assess the quality of the equilibrated residual method we compare three versions of the method. The differences between them result from the different definitions of the element local problems in these three versions. The collection of solutions to such problems constitutes the estimate of the exact global solution. In the first of the applied versions, we average the interelement stress fluxes. In the second case we perform linear (at the element vertex nodes) equilibration of these fluxes. In the third version we constrain local problems at element vertices by means of the displacement values obtained from the global numerical solution. Then, for the most effective case (the third one), we perform unique parametric studies of the modeling, approximation and total error estimations. These studies include such important factors as: the problem type, the applied mechanical model, and the mechanical complexity of the model. Our studies are completed with an analysis of the results. This analysis leads to practical hints concerning the appropriate definitions of the local problems, so as to assure the most effective error estimation within complex structures. Note that the numerical results presented here illustrate some of the tabular data published in [9].

## 2 MODIFICATION OF THE ERM FOR COMPLEX STRUCTURES

In order to apply the equilibrated residual methods (ERM) to the 3D-based models of complex structures, one has to include the following changes.

Firstly, the equilibration procedure has to be performed in the local nodal directions for the shell vertex nodes of the shell elements (see [7, 9]) and such nodes of the shell parts of the transition elements as well (compare [9]). These nodal directions are consistent with the shell mid-surface normal and tangent directions. In the normal direction different shape function definition has to be applied in the equilibration procedure so as to take into account the Reissner-Mindlin kinematic constraints of lack of elongation of the normals to the mid-surface. In the case of the solid nodes of the solid or hierarchical shell elements, and in such nodes of the solid or hierarchical shell parts of the solid-to-shell or shell-to-shell transition elements as well, the equilibration can be performed in the standard way [1, 3, 4], i.e. in the global directions. The standard 3D vertex shape functions are applied in this case.

And secondly, in the local ERM problems of the elements of the regular meshes, the nodal forces due to the equilibrated interelement stress fluxes have to be defined as global ones. In the case of the solid nodes, one can use the global splitting factors and global components of the interelement stresses for these forces determination. The applied shape functions are 3D vertex ones. On the contrary, in the case of the shell nodes, one has to utilize the local splitting factors and local components of the interelement stresses. One may apply the same shape functions as before, apart from the third local direction, where one has to apply the modified vertex shape functions, accounting for the Reissner-Mindlin

constraints. Finally, the products of the local factors and stresses have to be transformed to the global system of coordinates.

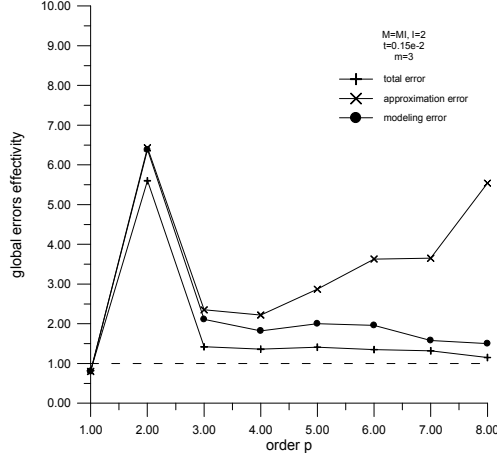
In the case of the irregular meshes, resulting from the local element subdivisions, one has to take into account that for the hanging (constrained) vertex nodes of the element obtained with the subdivision, the splitting factors are expressed through the corresponding factors of the vertex unconstrained (active) nodes of the undivided parent (or neighbouring) element. Examples of the corresponding relations, for the solid nodes, in the case of 2D- and 3D-problems, can be found in [4, 10] and [1], respectively. In this paper we extend this approach onto the shell nodes of the first-order shell and transition elements.

### 3 PARAMETRIC STUDIES OF THE ESTIMATORS

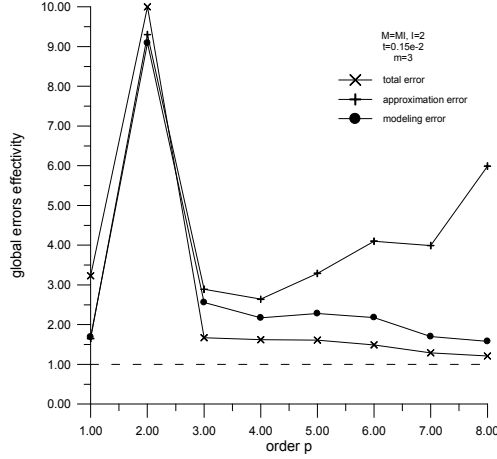
Here we take advantage the benchmark examples used by us elsewhere [1, 2, 9]. The first two examples concern a bending-dominated plate and a bending-dominated half-cylindrical shell. Both structures are aligned horizontally. The third example corresponds to a membrane-dominated cylindrical shell. The length of straight edges of the plate and shells are equal to  $2l = 3.14 \cdot 10^{-2}$  m, the curved edges of the shells are equal to  $2l = \pi r = 3.14 \cdot 10^{-2}$  m, with  $r = 1.0 \cdot 10^{-2}$  m. The thickness of the structures equals  $t = 0.15 \cdot 10^{-2}$  m. The plate is clamped. Also the straight edges of the first shell are clamped, while the curved ones are free. There is no rotation along the curved edges of the second shell. The two bending-dominated examples are loaded vertically with the uniform traction  $p = 4.0 \cdot 10^4$  N/m<sup>2</sup>, while the membrane-dominated shell is loaded with the internal pressure of the same magnitude. Due to the symmetry of the geometry, loading and boundary conditions we analyse only a quarter of the bending-dominated structures and one-eighth of the membrane-dominated one.

#### 3.1 Dependence of effectivities on the local problem definition

We analyze the residual method local problems of three types in this section. In figs. 1 and 2 we present the results obtained with averaging of the interelement stresses, for two definitions of the discretization parameters  $H$ ,  $P$  and  $Q$  in the local problems. Here  $H$ ,  $P$  and  $Q$  are the local problem counterparts of the global parameters  $h$ ,  $p$  and  $q$ . The first definition is:  $H = h$ ,  $P = p$ ,  $Q = q + 1$ , while the second one reads:  $H = h$ ,  $P = p + 1$ ,  $Q = q + 1$ . Subsequently, figs. 3 and 4 correspond to equilibration of the interelement tractions in the local problems for the two definitions, while figs. 5 and 6 correspond to constraining the vertex nodes of the elements with the values of displacements from the global problem. Note that in the first four cases the elements are constrained with six global displacements so as to remove three rigid body translations and three rigid body rotations. The presented results concern the approximation, modeling and total global error effectivity indices and correspond to the plate example. The calculations were performed for the hierarchical shell model  $MI$  of the second order ( $I \equiv q = 2$ ) and



**Figure 1:** Effectivity indices in the case of stress averaging ( $H = h$ ,  $P = p$ ,  $Q = q + 1$ )

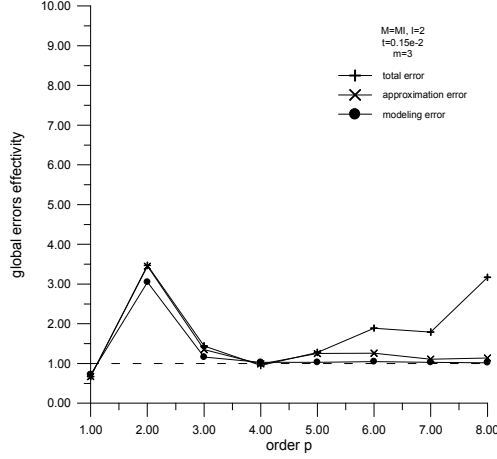


**Figure 2:** Effectivity indices in the case of stress averaging ( $H = h$ ,  $P = p + 1$ ,  $Q = q + 1$ )

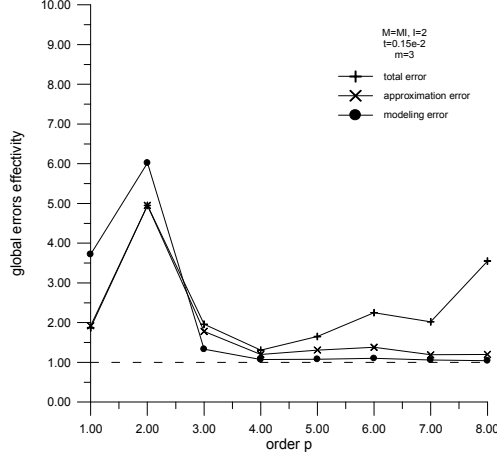
the uniform mesh division into  $2m^2$  prismatic elements ( $m \equiv l/h = 3$ ).

Analyzing all of the presented results, one can notice the worsening of the approximation error effectivity indices with low values of  $p$ . Also high values of  $p$  lead to worse values of the approximation error effectivities. This two observations can be related to the presence of the numerical locking for  $p = 2$  and the influence of the boundary layer phenomena for  $p \geq 6$ , respectively. Please note that the total and modelling error effectivities are affected by the locking phenomenon only.

The second observation is that, even though the equilibration delivers better effectivities than the averaging, neither the averaging nor the equilibration provide the satisfactory results of the residual-based global error estimation, as the effectivity indices are far above the desired value of 1.0. The best results are obtained for the constraining the local problems with the global displacements, with  $H = h$ ,  $P = p + 1$  and  $Q = q + 1$ . Because of that, our further numerical tests will be limited to this particular case.



**Figure 3:** Effectivity indices in the case of stress equilibration ( $H = h$ ,  $P = p$ ,  $Q = q + 1$ )

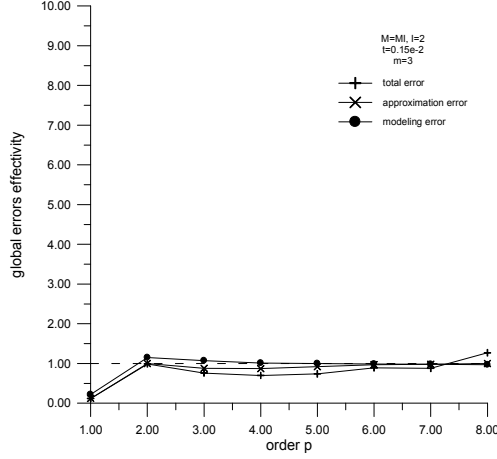


**Figure 4:** Effectivity indices in the case of stress equilibration ( $H = h$ ,  $P = p + 1$ ,  $Q = q + 1$ )

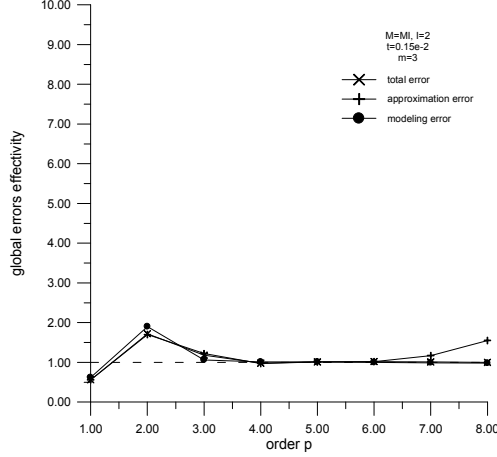
### 3.2 Influence of the problem type on effectivities

In order to determine the influence of the problem type (plate or shell, bending or membrane dominance) on effectivities we compare the corresponding results for the plate (the bending-dominated one), the bending-dominated shell, and the membrane-dominated shell as well. In the latter case we consider two problems, corresponding to hierarchical shell models  $MI$  of the first ( $I \equiv q = 1$ ) and second ( $I \equiv q = 2$ ) order, as for the membrane-dominated structures the improper solution limit phenomenon does not appear. In the case of the two bending-dominated examples only  $I \equiv q = 2$  is possible, because of this phenomenon appearance. In all examples we apply  $m = 4$ . The results, corresponding to the four respective cases, are presented in figs. 7, 8, 9 and 10.

Analyzing the results one can notice that for the bending- and membrane-dominated shell examples the approximation error effectivity is worse than for the plate. Only, the



**Figure 5:** Effectivity indices in the case of vertex constraints ( $H = h$ ,  $P = p$ ,  $Q = q + 1$ )



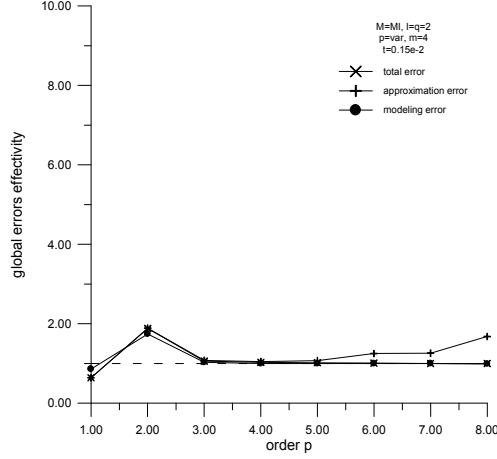
**Figure 6:** Effectivity indices in the case of vertex constraints ( $H = h$ ,  $P = p + 1$ ,  $Q = q + 1$ )

bending-dominated shell example is sensitive to the locking ( $p = 2, 3$ ) and boundary layer phenomena ( $p \geq 6$ ), in the way qualitatively similar to the plate example. Both membrane-dominated examples produce similar results. No influence of the locking and the boundary layer is observed.

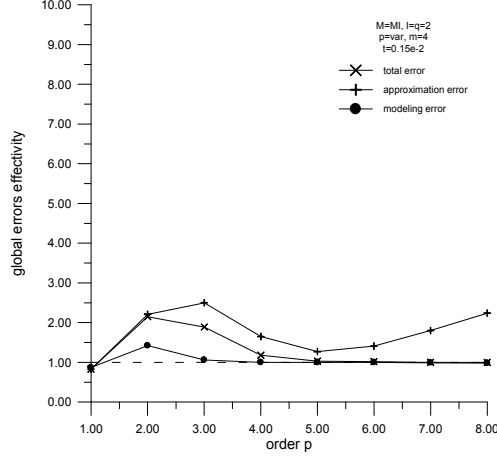
### 3.3 Influence of the model on effectivities

Here we compare the results presented in the previous subsection, corresponding to our three model problems and 3D-based hierarchical shell model  $MI$ , with the analogous results obtained for the 3D-based Reissner-Mindlin model  $RM$  of the plate and shells. The respective results for  $m = 4$ , in the case of the plate and bending- and membrane-dominated shells are shown in figs. 11, 12, 13, respectively.

It can be noticed that the  $RM$  model results are less sensitive to the locking phenomenon for the applied density of the mesh, due to better regularity of this model in



**Figure 7:** Plate problem effectivities versus the order  $p$  ( $M = M2$ ,  $t = 0.15 \cdot 10^{-2}$  m)



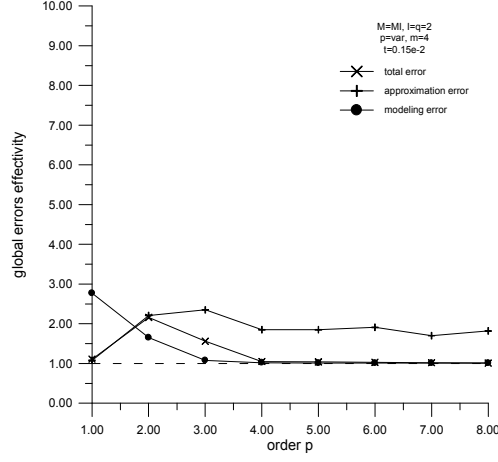
**Figure 8:** Half-cylindrical shell problem effectivities vs.  $p$  ( $M = M2$ ,  $t = 0.15 \cdot 10^{-2}$  m)

comparison to the hierarchical  $MI$  model. Moreover the influence of the boundary layers is not present for this model. This observation is consistent with the theory for the Reissner-Mindlin model.

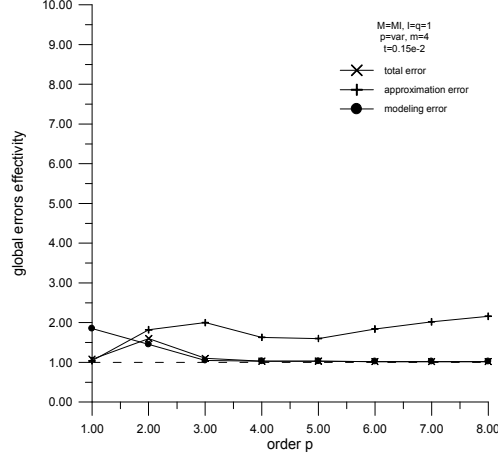
The second observation is that in the case of the two bending-dominated examples the total and modeling error effectivities are below 1.0 and equal to about 0.8.

### 3.4 Effectivities versus model complexity

In this subsection we introduce complex mechanical description of our plate and shell examples. This means that the hierarchical shell model  $MI$  ( $I \equiv q = 2$ ), the first-order shell model  $RM$  ( $q = 1$ ) and the 3D-based shell-to-shell transition model  $MI/RM$  ( $q = 1, 2$ ) are employed for each model structure. In the case of the plate, the  $RM$  square domain is symmetric and located in the interior of the plate. The lengths of a quarter of this domain are equal to  $l_{RM} = l/2$ . In the case of the bending-dominated half-cylindrical



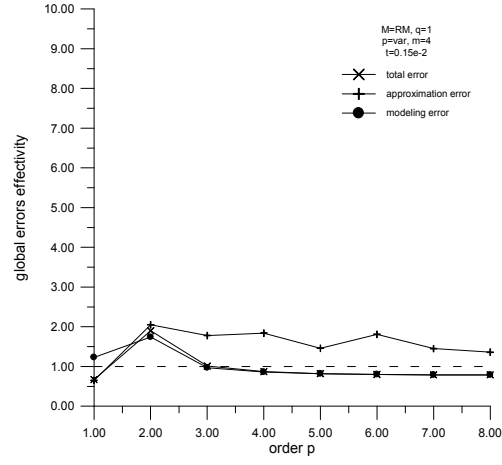
**Figure 9:** Cylindrical shell problem effectivities versus  $p$  ( $M = M2$ ,  $t = 0.15 \cdot 10^{-2}$  m)



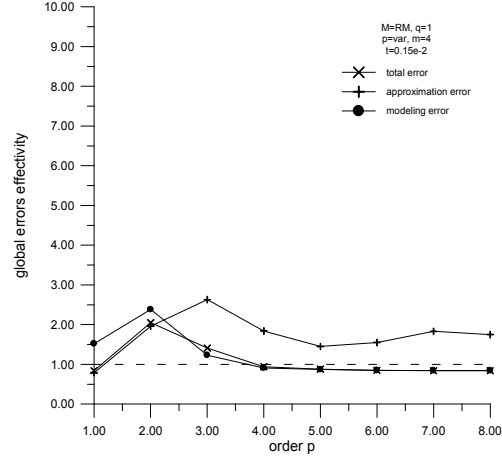
**Figure 10:** Cylindrical shell problem effectivities versus  $p$  ( $M = M1$ ,  $t = 0.15 \cdot 10^{-2}$  m)

shell, the  $MI$  zone is aligned along the straight clamped boundary of the shell. The curved boundary of a quarter of the shell is divided by two, i.e.  $l_{RM} = l/2 = \pi r/4$ . In the case of the membrane-dominated shell, the straight boundary of one-eighth of the shell is divided by two, i.e.  $l_{RM} = l/2$ , and the  $MI$  zone is aligned along the external curved boundary of the shell. The  $MI$  zones of the plate and shells are joined with the  $RM$  zones with one layer of the transition elements  $MI/RM$ , forming the transition domain  $TR$ .

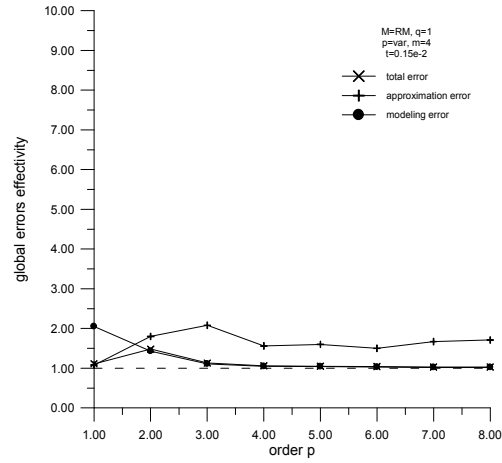
Comparing the complex models' results, presented in figs. 14, 15, 16 and 17, with the corresponding results for the pure  $MI$  models (figs. 7, 8, 9 and 10), one can see their close similarity. The only difference is that now, in the case of the two bending-dominated examples, the total and modeling error effectivities are slightly below 1.0, due to the presence of the  $RM$  zones in the complex models of the structures.



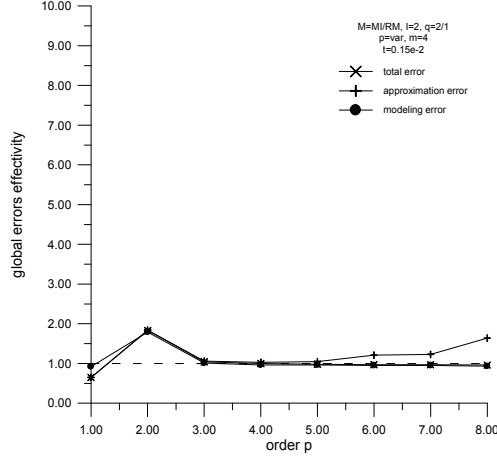
**Figure 11:** Plate problem effectivities versus the order  $p$  ( $M = RM$ ,  $t = 0.15 \cdot 10^{-2}$  m)



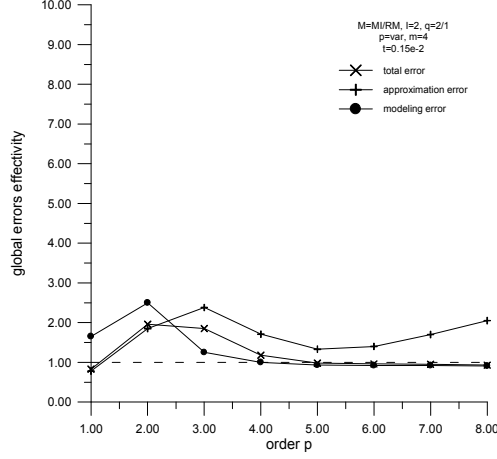
**Figure 12:** Half-cylindrical shell effectivities versus  $p$  ( $M = RM$ ,  $t = 0.15 \cdot 10^{-2}$  m)



**Figure 13:** Cylindrical shell problem effectivities versus  $p$  ( $M = RM$ ,  $t = 0.15 \cdot 10^{-2}$  m)



**Figure 14:** Plate problem effectivities versus  $p$  ( $TR = M2/RM$ ,  $t = 0.15 \cdot 10^{-2}$  m)

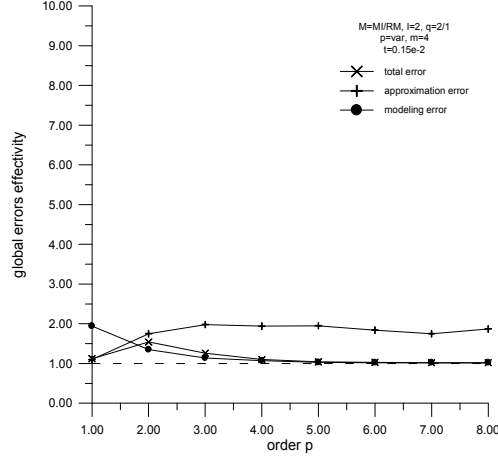


**Figure 15:** Half-cylindrical shell effectivities versus  $p$  ( $TR = M2/RM$ ,  $t = 0.15 \cdot 10^{-2}$  m)

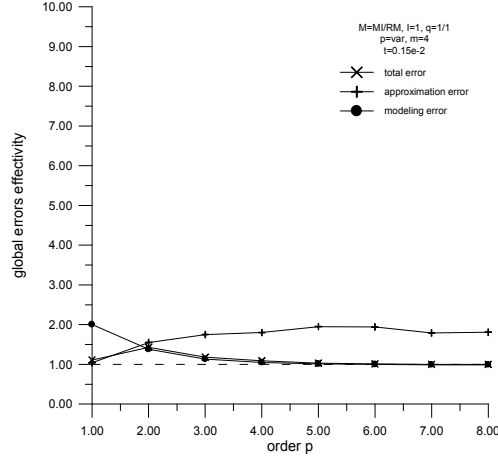
## 4 CONCLUSIONS

The theoretical findings, which concern the algorithms of the residual-based error estimation of the structures of 3D-based complex mechanical description, are as follows.

- The equilibration procedure, proposed in the case of the complex structures, needs distinction between the solid and shell vertex nodes, as in the latter case the application of the modified shape functions, which account for the Reissner-Mindlin constraints, is necessary. Also the application of the local directions, perpendicular and tangent to the shell mid-surface, is necessary in this case.
- The definitions of the vertex nodal forces, entering the ERM local problems and representing the equilibrated interelement stress fluxes, are also dependent on the vertex node type. In the case of the shell nodes, the local directions and the modified shape functions are applied again.



**Figure 16:** Cylindrical shell problem effectivities vs.  $p$  ( $TR = M2/RM$ ,  $t = 0.15 \cdot 10^{-2}$  m)



**Figure 17:** Cylindrical shell problem effectivities vs.  $p$  ( $TR = M1/RM$ ,  $t = 0.15 \cdot 10^{-2}$  m)

- In the case of the element local problems with hanging (constrained) nodes, the splitting functions and factors are expressed with the corresponding factors of the bigger undivided elements. The distinction between the solid and shell nodes may be in use again, i.e. both the local and global splitting factors may enter the calculation of the forces acting in the element vertex hanging nodes. Such complex situations happen for the  $h$ -refined transition elements.

The conclusions, concerning parametric studies of the error estimation with the element residual methods, can be formulated as follows.

- The version of the element residual method based on the constraints defined with the global problem vertex displacements, with  $H = h$ ,  $P = p + 1$ ,  $Q = q + 1$  applied in the local element problems, delivers the effectivity results closest to the desired value of 1.0. This version is better than the two approaches based on the averaging

or equilibration of the interelement stresses in the local problems.

- In the case of this constrained version of the ERM local problems, all three global effectivities are above 1.0 for the purely hierarchical models *MI* of the structures. In the case of the bending-dominated *RM* and complex models of the structures, the total and modelling error effectivities are below 1.0.

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