ADAPTIVE LIMIT ANALYSIS USING DEVIATORIC FIELDS

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Abstract. Accurate estimates of limit loads for difficult stability problems in geotechnical engineering can rarely be obtained from a single finite element limit analysis without using an excessive number of elements. Therefore, efficient adaptive strategies which maximize the solution accuracy using minimum number of elements in the mesh are of great interest. This study explores the possibility of using the internal dissipation calculated from deviatoric stresses and strain rates as suitable control field for purely frictional materials. The performance observed for considered set of problematic for other adaptive schemes geotechnical examples is very promising. Moreover, the proposed approach works very well also for cohesive and cohesive frictional materials, suggesting its use as general engine for adaptive mesh refinement.

1 INTRODUCTION

For complex, practical stability problems in geotechnical engineering, accurate estimates of the collapse load or factor of safety can rarely be obtained from a single analysis and a trial and error process is usually required. The key to obtaining accurate solutions lies in accurately capturing the areas of plasticity within the problem domain, as their pattern and intensity govern the solution. The development of an efficient mesh adaptivity strategy, which is able to pinpoint the fine detail of a structure's collapse mechanism, is thus of the highest priority in modern limit and shakedown analysis.

A critical aspect of any adaptive meshing process is the estimation of the discretisation error present in a given finite element solution. Since *a priori* error estimates play only an indicative role (Borges *et al.*^[1]), useful error estimates must employ *a posteriori* techniques to predict the overall discretisation error in one or more solution norms (or control variables). Generally speaking, two major approaches have been practiced so far. The first is the Hessian based error estimation, where the spatial distribution of the error in solution is obtained on the basis of information gathered from the matrix of second derivatives of some control variable (Zienkiewicz *et al.*^[2], Almeida *et al.*^[3], Lyamin *et al.*^[4]). And the second is a so-called *gap* adaptivity scheme, which is based on the fact that for limit analysis applications the global error in the solution can be readily obtained as the sum of elemental differences between upper and lower bound estimates (Ciria *et al.*^[5], Muñoz *et al.*^[6]).

The major advantage of Hessian based error estimation when combined with optimalmesh-adaptive scheme is that it usually provides the element size distribution which converges (keeping the number of elements in the mesh constant) very quickly to a steady, smoothly graded mesh pattern, which can be either isotropic or anisotropic. It is very general and based on the fact that, at some point **x** in the vicinity of a point \mathbf{x}_0 , the difference between the variable of interest **u** and its discrete approximation \mathbf{u}_h can be estimated using the following expression

$$\left\|\mathbf{u} - \mathbf{u}_{h}\right\| \simeq C \left\| (\mathbf{x} - \mathbf{x}_{0})^{T} \mathbf{H}_{R}(\mathbf{u}_{h}(\mathbf{x}_{0}))(\mathbf{x} - \mathbf{x}_{0}) \right\|$$
(1)

where *C* is a positive constant and $\mathbf{H}_{R}(\mathbf{u}_{h}(\mathbf{x}_{0}))$ denotes a recovered Hessian matrix. An anisotropic error estimator for element *e* of a partition \mathcal{T}_{h} of the domain Ω can then be introduced as

$$\eta_e = n\Omega_e^{1/2} \left| \lambda_n(\mathbf{x}_0) \right| h_n^2 \geq \left\{ \int_{\Omega_e} \left((\mathbf{x} - \mathbf{x}_0)^T \mathbf{H}_R(\mathbf{u}_h(\mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0) \right)^2 \mathrm{d}\Omega \right\}^{1/2}; \qquad \eta = \sum_e \eta_e \tag{2}$$

where *n* is the problem dimensionality, h_n is the minimum dimension of element *e*, and λ_n is the largest eigenvalue of the element Hessian matrix. It is assumed also that the estimated error yields the same value in any direction, i.e. $|\lambda_1| h_1^2 = |\lambda_2| h_2^2 = \dots = |\lambda_n| h_n^2$.

The choice of a suitable control variable is not obvious for plasticity problems. Several approaches have been practiced so far including those based on power dissipation or its gap (Ciria *et al.*^[5], Muñoz *et al.*^[6]), plastic multipliers (Lyamin *et al.*^[4]) and strain rate (Christiansen & Pedersen^[7]) fields employed as control variables. All these schemes work quite well for cohesive or cohesive-frictional materials, but for purely frictional soils their performance stalls as e.g. plastic multipliers have substantially high values for all zero stress points on the surface of soil domain, therefore cannot indicate reliably plastic areas. Similar conclusion can be made about performance of schemes based on power dissipation or strain rates. This study explores the possibility of using the internal dissipation calculated from deviatoric stresses and strain rates (called also "shear power" in the rest)

$$u = \int_{\Omega_{e}} \mathbf{s} : \dot{\mathbf{\varepsilon}}' \mathrm{d}\Omega, \quad s_{ij} = \sigma_{ij} - \frac{1}{3} I_{1}^{\sigma} \delta_{ij}, \quad \dot{\varepsilon}'_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3} I_{1}^{\dot{\varepsilon}} \delta_{ij}$$
(3)

as suitable control field for purely frictional materials. In above σ_{ij} , s_{ij} and $\dot{\varepsilon}_{ij}$, $\dot{\varepsilon}'_{ij}$ are the Cartesian and deviatoric stresses and strain rates, respectively, and I_1^{σ} , $I_1^{\dot{\varepsilon}}$ are the first invariants for stresses and strain rates.

The performance observed for considered set of problematic for other adaptive schemes geotechnical examples is very promising. Moreover, the proposed approach works very well also for cohesive and cohesive frictional materials, suggesting its employment as general engine for adaptive mesh refinement.

2 THE OPTIMAL MESH ADAPTIVE SCHEME

Usually mesh refinement proceeds with gradual adjustment of the element size aiming to distribute local error uniformly over the problem domain. The other alternative is to obtain the

element size distribution which minimizes the global error given by equation (2). This approach is known as optimal-mesh-adaptive technique and is described in detail e.g. by Almeida *et al.*^[3] In brief, the optimal-mesh-adaptive procedure can be cast as constrained optimization problem, which for two-dimensional case becomes

minimise
$$\mathcal{F}(h_{2T}) = \{\eta(h_{2T})\}^p = \sum_{T \in \mathcal{T}_k} \Omega_T |2\lambda_{2T}|^p h_{2T}^{2p}$$

subject to $N_e = (4/\sqrt{3}) \sum_{T \in \mathcal{T}_k} \Omega_T / (s_T h_{2T}^2)$ to find $h_{2T}, T \in \mathcal{T}_k$ (4)

where h_{2T} and s_T are the new size and the stretching of element *T*, \mathcal{T}_k is the finite element discretization at the adaptation step *k* and N_e is the desired number of elements at the step *k*+1. For p = 2 and the case of equilateral elements (no stretching) the solution to problem (4) is given by

$$h_{2T} = \sqrt{\left(4\Omega_e / \left(\sqrt{3}N_e \eta_e\right)\right)\eta} \tag{5}$$

The advancing front algorithm (Peraire *et al.*^[8]) has been employed for generating the mesh. As the meshing time is only a small fraction of the total CPU time in adaptive limit analysis, this algorithm was chosen in order to give full control of the mesh quality, including the shape of the elements and the rate of change of the element size throughout the mesh from one iteration to the next. Both refinement and coarsening of the mesh have been allowed.

3 LIMIT ANALYSIS

The lower (LB) and upper (UB) bound limit analysis formulations used in this investigation stem from the methods originally developed by Sloan^{[9][10]}, but have evolved significantly over the past two decades to incorporate the major improvements described in Lyamin and Sloan^{[11][12]} and Krabbenhoft et al.^{[13][14]}. Key features of the methods include the use of linear finite elements to model the stress/velocity fields, and collapsed solid elements at all inter-element boundaries to simulate stress/velocity discontinuities. The solutions from the lower bound formulation yield statically admissible stress fields, while those from the upper bound formulation furnish kinematically admissible velocity fields. This ensures that the solutions preserve the important bounding properties of the limit theorems.

Both formulations result in convex mathematical programs, which (considering the dual form of upper bound problem) can be cast in the following form:

maximize
$$\lambda$$

subject to $\mathbf{A}\boldsymbol{\sigma} = \mathbf{p}_0 + \lambda \mathbf{p}$ (6)
 $f_i(\boldsymbol{\sigma}) \le 0, \quad i = \{1, \dots, N\}$

where λ is a load multiplier, σ is a vector of stress variables, **A** is a matrix of equality constraint coefficients, \mathbf{p}_0 and \mathbf{p} are vectors of prescribed and optimizable forces, respectively, f_i is the yield function for stress set *i* and *N* is the number of stress nodes. The

solutions to problem (6) can be found efficiently by using general Interior-Point methods (IPM) or specialised conic optimization solvers (SOCP).

4 NUMERICAL EXAMPLES

Two representative examples from the soil mechanics are considered in this section to illustrate the efficiency of proposed adaptive approach. First example, so-called N_{γ} problem, is about estimating the bearing capacity of rigid footing resting on cohesionless soil (sand). Second example is known as "passive earth pressure" case. Here the maximum lateral pressure which can be exerted to the soil cut, before it collapses upwards, needs to be found. Both examples are treated as two-dimensional problems and considered under plain strain conditions.

Adaptive refinement proceeds by specifying the initial and target number of elements in the mesh, and the number of adaptive iterations. If this target number of elements is reached before the maximum number of iterations has exceeded, no additional elements are injected. However, some improvement can still be achieved by redistributing the element sizes in the remaining iterations if a better pattern of the control variable can be found. In examples considered the thresholds on mesh refinement and coarsening factors between 2 iterates were set to 0.25 and 1.5, respectively.

4.1 Rigid rough strip footing on cohesionless soil (N_{γ} problem)

For a rigid strip footing resting on a ponderable purely frictional soil with no surcharge the bearing capacity is usually estimated by using reduced Terzaghi^[15] equation of the form

$$q = 0.5\gamma BN_{\gamma} \tag{7}$$

where γ is soil unit weight, *B* is the width of the footing and N_{γ} is the bearing capacity factor, which depends on soil friction angle, ϕ . There is no exact solution available for N_{γ} and over the years several empirical expressions were suggested and used in practice (Brinch Hansen^[16], Caquot & Kerisel^[17]). Recently very accurate estimates done by numerical limit analysis were reported (Hjiaj *et al.*^[18]) and eventually quasi-exact values of N_{γ} were obtained by the method of characteristics (Martin^[19]). Therefore, besides the standard for limit analysis UB-LB gap error estimation, this allows direct check of the accuracy of adaptively obtained



Figure 1. Geometry (a), initial mesh (b) and shear power dissipation plot (c) for strip footing.

solutions for this problem.

The problem description (including Prandtl^[20] failure mechanism) together with the initial mesh used for analysis and corresponding shear power dissipation is given in Figure 1. Next, in Figure 2 the distributions of several traditionally used for adaptive limit analysis control variables are plotted. Due to the absence of cohesion in soil mass it is evident that power dissipation (all zeros) is not an alternative to govern refinement procedure in this case. And



Figure 2. Distributions of commonly used control variables in the case of N_{γ} problem.

even if power loss due to soil unit weight is taken into account (Figure 2b) the resultant distribution does not resemble the actual collapse mechanism (slip line) to be considered as a good choice. Neither it will work when UB-LB gap of elemental power dissipation would be used. Similar comments are applied to another pair of control variables, strain rate and plastic multiplier fields. It is clear that all zero-stress points (soil surface boundary, LB case) are at plastic state, therefore will have some non-zero plastic multipliers as shown in (Figure 2d). This "noise" prevents plastic multipliers to be employed as adaptivity guide either. On the other hand, the dissipation computed using deviatoric terms of stresses and strains (shear power) has very distinctive distribution resembling classical Prandtl^[20] collapse mechanism for strip footing. And, as can be judged from results presented in Figure 3c, it works efficiently for both lower and upper bound discretizations. The final mesh and corresponding shear power dissipation are illustrated in Figure 3a,b.



Figure 3. Final mesh (a), shear power dissipation (b) and convergence diagram for strip footing.

4.2 Passive earth pressure

This is another classical problem in soil mechanics, where the lateral pressure, p, is applied to the soil mass to cause its collapse, as shown in Figure 4. There are several theories for this problem (the most famous are due to Coulomb^[21] and Rankine^[22]) with different analytical solutions accounting for various soil slope angles, soil/wall interface conditions, mode of



Figure 4. Geometry (a), initial mesh (b) and shear power dissipation plot (c) for passive earth pressure.

failure (no rotation or rotation allowed), etc. But our main focus here is not actually to compare results obtained to existing solutions, rather demonstrate that proposed mesh



Figure 5. Distributions of commonly used control variables in the case of passive earth pressure problem.

refinement approach performs reliably when applied to sands. For this purpose, in the same way as for N_{γ} case, the distributions of most popular control variables traditionally used within the limit analysis adaptive schemes are given in Figure 5. It appears that the same comments as those given in previous section are applicable here as well - none of the distributions in Figure 5 seems to be suitable to assist with effective mesh refinement. On the other hand, using proposed adaptive scheme based on shear power dissipation results in robust refinement procedure as presented in Figure 6.



Figure 6. Final mesh (a), shear power dissipation (b) and convergence plot (c) for passive earth pressure problem.

5 CONCLUSIONS

Based on deviatoric stress and strain fields elemental power dissipation was employed to control mesh refinement process in limit analysis computations for purely frictional materials. Both lower and upper bound counterparts of limit analysis were tested. The obtained results show that the proposed approach works reliably for demanding applications, where traditionally used control variables fail to perform.

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