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A PRIORI BASED MESH ADAPTATION FOR VISCOUS FLOW

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Abstract. A priori estimates are applied to the anisotropic mesh adaptation for 2D viscous flows and 3D ones including Large Eddy Simulation.

1 METHODS

Two novelties were presented [3] in the recent Eccomas conference at Vienna. First, an a priori analysis for diffusive flows allowing, through a goal-oriented criterion, the direct specification of mesh metric, *i.e.* mesh stretching and density, [6]. It is an extension of the mesh adaptation technique refered as the global unsteady fixed point algorithm. In [5] this algorithm was applied to Euler flows. This algorithm involves the following ingredients: (1) an *a priori* goal oriented error estimate based on an adjoint allowing to define an optimal metric at each time level, (2) a fixed point encapsulating a time advancing loop, a backward loop for adjoint, and the generation of a sequence of adapted meshes. Second, this method is extended to the reduction of approximation error in LES formulations [4]. We define the convergence of the LES discrete model to a continuous filtered PDE with prescribed turbulent viscous term and show that an optimal mesh can be defined according to the goal-oriented optimal metric theory. With the help of the above fixed-point algorithm, this gives an optimal mesh for a prescribed filter. This process is then equipped of an external loop for computing the filter as an LES one. These methods were in [3] benchmarked with the 3D turbulent flow around a cylinder (Reynolds number= 3900). In complement to this very preliminary numerical experiment, we discuss here an industrial test case related to offshore platforms.

2 Continuous mesh theory

2.1 Mesh parametrization

We propose to work in the continuous mesh framework, defined in [1, 2]. The main idea of this framework is to model discrete meshes by Riemannian metric fields. It allows us to define proper differentiable optimization *i.e.,* to use a calculus of variations on continuous metrics which cannot apply on the class of discrete meshes. This framework lies in the class of metric-based methods. A continuous mesh $\mathcal{M}$ of the computational domain Ω is identified to a Riemannian metric field $\mathcal{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$. For all $\mathbf{x}$ of Ω, $\mathcal{M}(\mathbf{x})$ is a symmetric 3×3 matrix having $(\lambda_i(\mathbf{x}))_{i=1,3}$ as eigenvalues along the principal directions $\mathcal{R}(\mathbf{x}) = (\mathbf{v}_i(\mathbf{x}))_{i=1,3}$. Sizes along these directions are denoted $(h_i(\mathbf{x}))_{i=1,3} = (\lambda_i^{-\frac{1}{2}}(\mathbf{x}))_{i=1,3}$ and the three *anisotropy quotients* r_i are defined by: $r_i = h_i^3\,(h_1h_2h_3)^{-1}$. The diagonalisation of $\mathcal{M}(\mathbf{x})$ writes:

$$\mathcal{M}(\mathbf{x}) = d^{\frac{2}{3}}(\mathbf{x})\,\mathcal{R}(\mathbf{x}) \begin{pmatrix} r_1^{-\frac{2}{3}}(\mathbf{x}) & & \\ & r_2^{-\frac{2}{3}}(\mathbf{x}) & \\ & & r_3^{-\frac{2}{3}}(\mathbf{x}) \end{pmatrix} {}^t\mathcal{R}(\mathbf{x}), \tag{1}$$

The *vertex density* d is equal to: $d = (h_1h_2h_3)^{-1} = (\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}} = \sqrt{\det(\mathcal{M})}$. By integrating it, we define the *total number of vertices* $\mathcal{C}$:

$$\mathcal{C}(\mathcal{M}) = \int_\Omega d(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Omega \sqrt{\det(\mathcal{M}(\mathbf{x}))}\,\mathrm{d}\mathbf{x}.$$

Given a continuous mesh $\mathcal{M}$, we shall say, following [1, 2], that a discrete mesh $\mathcal{H}$ of the same domain Ω is a **unit mesh with respect to** $\mathcal{M}$, if each tetrahedron $K\in\mathcal{H}$, defined by its list of edges $(\mathbf{e}_i)_{i=1...6}$, verifies:

$$\forall i \in [1,6], \quad \ell_\mathcal{M}(\mathbf{e}_i) \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right] \quad \text{and} \quad Q_\mathcal{M}(K) \in [\alpha, 1] \ \text{ with } \ \alpha > 0\,,$$

in which the length of an edge $\ell_\mathcal{M}(\mathbf{e}_i)$ and the quality of an element $Q_\mathcal{M}(K)$ are defined as follows:

$$\ell_\mathcal{M}(\mathbf{e}_i) = \int_0^1 \sqrt{{}^t\mathbf{ab}\;\mathcal{M}(\mathbf{a}+t\,\mathbf{ab})\;\mathbf{ab}}\,\mathrm{d}t, \quad \text{with } \mathbf{e}_i = \mathbf{ab},$$

$$Q_\mathcal{M}(K) = \frac{36}{3^{\frac{1}{3}}} \frac{\left(\int_K \sqrt{\det(\mathcal{M}(\mathbf{x}))}\,\mathrm{d}\mathbf{x}\right)^{\frac{2}{3}}}{\sum_{i=1}^6 \ell^2_\mathcal{M}(\mathbf{e}_i)} \in [0,1].$$

We choose a tolerance α equal to 0.8. We want to emphasize that the set of all the discrete meshes that are unit meshes with respect to a unique $\mathcal{M}$ contains an infinite number of meshes.

2.2 Continuous interpolation error

Given a smooth function u, to each unit mesh $\mathcal{H}$ with respect to $\mathcal{M}$ corresponds a local interpolation error $|u-\Pi_{\mathcal{H}}u|$. In [1, 2], it is shown that all these interpolation errors are well represented by the so-called continuous interpolation error related to $\mathcal{M}$, which is locally expressed in terms of the Hessian H_u of u as follows:

$$\begin{aligned}|u-\pi_{\mathcal{M}}u|(\mathbf{x},t) &= \frac{1}{10}\mathrm{trace}(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x})\,|H_u(\mathbf{x},t)|\,\mathcal{M}^{-\frac{1}{2}}(\mathbf{x}))\\ &= \frac{1}{10}d(\mathbf{x})^{-\frac{2}{3}}\sum_{i=1}^{3} r_i(\mathbf{x})^{\frac{2}{3}}\,{}^t\mathbf{v}_i(\mathbf{x})\,|H_u(\mathbf{x},t)|\,\mathbf{v}_i(\mathbf{x}),\end{aligned} \tag{2}$$

where $|H_u|$ is deduced from H_u by taking the absolute values of its eigenvalues and where time-dependency notations “$,t)$” have been added for use in next sections.

3 Mesh adaptation for laminar flow

We write in short the Navier-Stokes equations as follows:

$$\Psi(W)=0 \quad \text{with} \quad \Psi(W)=\frac{\partial W}{\partial t}+\nabla\cdot\mathcal{F}(W) \quad + \quad \textit{boundary conditions} \tag{3}$$

where notation $\mathcal{F}(W)$ involves the classical inviscid and viscous fluxes. We are interested in expressing the approximation error of a functional

$$j \;=\; (g,W)$$

depending on the unknown state W, in terms of interpolation error for functions of the state, weighted by derivatives of the adjoint. The continuous adjoint system related to the objective functional writes:

$$W^*\in\mathcal{V}\ ,\ \forall\psi\in\mathcal{V}\ :\ \left(\frac{\partial\Psi}{\partial W}(W)\psi,W^*\right)-(g,\psi)\;=\;0. \tag{4}$$

From Functional Analysis standpoint, a well-posed continuous adjoint system can be derived for any functional output as far as the linearized system is well posed. The discrete adjoint systeme writes:

$$W_h^*\in\mathcal{V}_h\ ,\ \forall\phi_h\in\mathcal{V}_h\ :\ \left(\frac{\partial\Psi_h}{\partial W}(W_h)\phi_h,W_h^*\right)-(g,\phi_h)\;=\;0. \tag{5}$$

The error on functional can be written:

$$\delta j = (g, \Pi_h W - W_h) \approx (W_h^*, \Psi(W) - \Psi_h(\Pi_h W)).$$

The method proposed here involves some heuristics. Indeed, we assume that the interpolate of the adjoint is close to the discrete adjoint:

$$\Pi_h W^* \approx W_h^*. \tag{6}$$

Therefore:

$$\delta j \approx (\Pi_h W^*, \Psi(W) - \Psi_h(\Pi_h W)).$$

According to the *a priori estimate* established in [6], we have:

$$\delta j \approx \sum_{mn} \int_\Omega \; G_{m,n}(W, \nabla W_m^*, \lambda(W_m^*)) \;\; |S_{m,n}(W) - \Pi_h S_{m,n}(W)| \;\; dv.$$

where $G_{m,n}$ is a function of W, ∇W^*, and $\lambda(W^*)$, maximal eigenvalue of W^*'s Hessian, and $S_{m,n}$ depends only on W.

A continuous error model is derived by replacing the mesh by a metric $\mathcal{M}$ and the interpolation error by $1 - \pi_{\mathcal{M}}$ by the continuous analog, as introduced in Section 2:

$$\delta j \approx \sum_{mn} \int_\Omega G_{m,n}(W, \nabla W_m^*, \lambda(W_m^*))|(1 - \pi_h)\; S_{m,n}(W)|\; dv.$$

Let us define the positive symmetric matrix

$$\mathbf{H}(\mathbf{x}, t) = \sum_{m,n} \; G_{m,n}(W, \nabla W_m^*, \lambda(W_m^*)) \; |H_{S_{m,n}(W)}| \tag{7}$$

where $|H_{S_{m,n}(W)}|$ holds for the absolute value of the Hessian matrix of function $S_{m,n}(W)$. Then we are interested into minimising the following error model:

$$\mathbf{E}(\mathcal{M}) \;\; = \;\; \int_0^T \int_\Omega \mathrm{trace}\left(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x}, t)\, \mathbf{H}(\mathbf{x}, t)\, \mathcal{M}^{-\frac{1}{2}}(\mathbf{x}, t)\right) \mathrm{d}\Omega\, \mathrm{d}t$$

The mesh optimization problem writes:

$$\text{Find } \mathcal{M}_{opt} = \mathrm{Argmin}_{\mathcal{M}}\, \mathbf{E}(\mathcal{M}), \tag{8}$$

under the constraint of bounded mesh fineness:

$$\mathcal{C}_{st}(\mathcal{M}) = N_{st}, \tag{9}$$

where N_{st} is a specified total number of nodes. Since we consider an unsteady problem, the space-time (st) complexity used to compute the solution takes into account the time discretization. The above constraint then imposes the total number of nodes in the time integral, that is:

$$\mathcal{C}_{st}(\mathcal{M}) = \int_0^T \tau(t)^{-1} \left(\int_\Omega d_{\mathcal{M}}(\mathbf{x}, t) \mathrm{d}\mathbf{x} \right) \mathrm{d}t \tag{10}$$

where $\tau(t)$ is the time step used at time t of interval $[0, T]$.

Let us assume that at time t, we seek for the optimal continuous mesh $\mathcal{M}_{go}(t)$ which minimizes the instantaneous error, *i.e.,* the spatial error for a fixed time t:

$$\tilde{\mathbf{E}}(\mathcal{M}(t)) = \int_\Omega \operatorname{trace}\left(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x}, t)\, \mathbf{H}(\mathbf{x}, t)\, \mathcal{M}^{-\frac{1}{2}}(\mathbf{x}, t) \right) \mathrm{d}\mathbf{x}$$

under the constraint that the number of vertices is prescribed to

$$\mathcal{C}(\mathcal{M}(t)) = \int_\Omega d_{\mathcal{M}(t)}(\mathbf{x}, t)\, \mathrm{d}\mathbf{x} = N(t). \tag{11}$$

Solving the optimality conditions provides the *optimal goal-oriented ("go") instantaneous continuous mesh* $\mathcal{M}_{go}(t) = (\mathcal{M}_{go}(\mathbf{x}, t))_{\mathbf{x}\in\Omega}$ at time t defined by:

$$\mathcal{M}_{go}(\mathbf{x}, t) = N(t)^{\frac{2}{3}}\, \mathcal{M}_{go,1}(\mathbf{x}, t)\,, \tag{12}$$

where $\mathcal{M}_{go,1}$ is the optimum for constraint $\int_\Omega d_{\mathcal{M}}(\mathbf{x}, t)\mathrm{d}\mathbf{x} = 1$:

$$\mathcal{M}_{go,1}(\mathbf{x}, t) = \left(\int_\Omega (\det \mathbf{H}(\bar{\mathbf{x}}, t))^{\frac{1}{5}} \mathrm{d}\bar{\mathbf{x}} \right)^{-\frac{2}{3}} (\det \mathbf{H}(\mathbf{x}, t))^{-\frac{1}{5}}\ \mathbf{H}(\mathbf{x}, t). \tag{13}$$

The corresponding optimal instantaneous error at time t writes:

$$\tilde{\mathbf{E}}(\mathcal{M}_{go}(t)) = 3\, N(t)^{-\frac{2}{3}} \left(\int_\Omega (\det \mathbf{H}(\mathbf{x}, t))^{\frac{1}{5}} \mathrm{d}\mathbf{x} \right)^{\frac{5}{3}} = 3\, N(t)^{-\frac{2}{3}}\, \mathcal{K}(t) \tag{14}$$

with $\mathcal{K}(t) = \left(\int_\Omega (\det \mathbf{H}(\mathbf{x}, t))^{\frac{1}{5}} \mathrm{d}\mathbf{x} \right)^{\frac{5}{3}}$. As in [5] (in which details can be found, the space-time problem is then solved by optimising $N(t)$ under the conditions that:

$$\int_0^T \tau(t)^{-1}\ N(t)\ \mathrm{d}t \ = \ N_{st}.$$

The unsteady optimality system (3,4,13) is solved by applying the global unsteady fixed-point adaptation algorithm introduced in [5]. Figures from an application to the impulsive start around an airfoil (Reynolds 1000) are depicted in Figure 1 which show meshes an flow density at three time levels of the adaptive calculation.

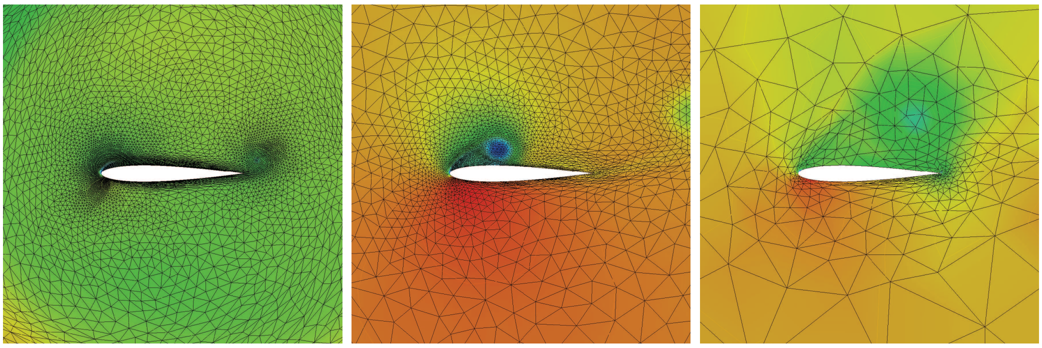

Figure 1: Impulsive start around an airfoil, density and mesh at three times

4 Mesh adaptation for turbulent flow

Convergence of LES models is a complicated issue since their filter depends of mesh size. As a consequence, when h tends to zero, LES comes closer to DNS, a positive behavior which, however does not give any chance to mesh convergence. Returning to a finite size mesh practical application, it is anticipated that the solution is made of filtered small scales and of well-resolved larger scales. In larger scales, we mean quasi-steady large scales and fluctuating intermediate scales, and both need accurate resolution. Let us consider the case where LES is obtained by adding as filter a Boussinesq term to the Navier-Stokes model. The LES model can be built on a modern model like WALE, which, in contrast to the classical Smagorinsky model, does not suffer of spuriously large dissipation in boundary layers, [8]. This can also be built by a dynamic version of the Smagorinsky model. This class of LES models can be described as discretizations of the *continuous filtered Navier-Stokes equations* for $W = ()$ built from the combination of the Navier-Stokes equation with a filter term and which we write in short:

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathcal{F}(W) = -\tau^{LES}(W). \tag{15}$$

The Boussinesq term $\tau^{LES}(< W >)$ writes:

$$\tau^{LES}(W) = \nabla \cdot \ \mu_{\mathrm{sgs}} \nabla \begin{pmatrix} 0 \\ u \\ v \\ 0 \end{pmatrix} \quad \text{with} \quad \mu_{\mathrm{sgs}} = \rho \left(C_{\mathrm{s}}\Delta\right)^2 \left|\widetilde{S}\right| \quad , \tag{16}$$

and is weighted by a scalar field, the local filter width $\Delta = \Delta(\mathbf{x}, t)$. C_{s} is the Smagorinsky coefficient (in practice we shall use the dynamic one, which is a function of W) and

$$\left|\widetilde{S}\right| = \sqrt{2\widetilde{S_{ij}}} \quad \text{with} \quad \widetilde{S_{ij}} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right).$$

As already mentioned, a LES model is a discretization of (15) and needs to be computed on a mesh. Then the best predictivity could be classically attained when the local filter width is taken equal to local mesh size.

Instead, we keep some more time the continuous formulation, and we consider the case where the local filter size is prescribed. It is prescribed as a given continuous scalar field. We call filtered continuous model the Navier-Stokes model with the extra Boussinesq term relying on the continuous filter size:

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathcal{F}(W) = -\nabla \cdot \ \mu(\Delta)\nabla \begin{pmatrix} 0 \\ u \\ v \\ 0 \end{pmatrix} \quad \text{with} \quad \mu(\Delta) = \rho\,(C_s\Delta)^2\left|\widetilde{S}\right|, \tag{17}$$

the solution of which is denoted W_Δ.

Given an discrete approximation of (18) which produce a solution $W_\Delta(\mathcal{M})$ on a mesh $\mathcal{M}$, our concern is now the following problem:

For a prescribed Δ, find a mesh that is therefore independant of Δ, of a given number of nodes, which minimizes:

$$|\mathcal{E}_\Delta(\mathcal{M})| = |W_\Delta - W_\Delta(\mathcal{M})|.$$

Symbols $\mathcal{E}_\Delta(\mathcal{M})$ define the *weak error*, *i.e.* the deviation between the discrete LES and its continuous analog, both defined for the given (mesh independant) filter width. A basic choice for the norm is an integral on a time interval $0, T$ of the L^1 spatial norm of the instantaeous deviation.

$$|\mathcal{E}_\Delta(\mathcal{M})| = |\int_0^T \int_\Omega W_\Delta - W_\Delta(\mathcal{M})\ d\mathbf{x}\ dt|.$$

Given now a number of nodes, the optimal mesh for reducing the weak error on a scalar output can be obtained by applying our global unsteady fixed point method with a prescribed filter size. Thanks to the global unsteady fixed point adaptation algorithm, the mesh adaptation concentrates resolution on unsteady turbulent structures.

Δ strategies. The above optimal mesh depends on the parameter Δ. In fact we want the best mesh $\mathcal{M}_{opt}$ for the Δ which is the local mesh size of this optimal mesh. We can solve this by an external fixed point.

5 An example

The proposed mesh adaptation method is applied to the computation of the flow around an offshore platform with a very complex geometry. This flow was accurately computed and compared with experiments in a specialized conference [7]. For the present mesh adaptive calculation, we take into account a large enough time interval and compute the adjoint on this interval. The resulting mesh adaptation criterion can be concentrated on the generation of a single mesh, since the vortices concentrate on a region of wake which is well identified by the algorithm. See Figure 2.

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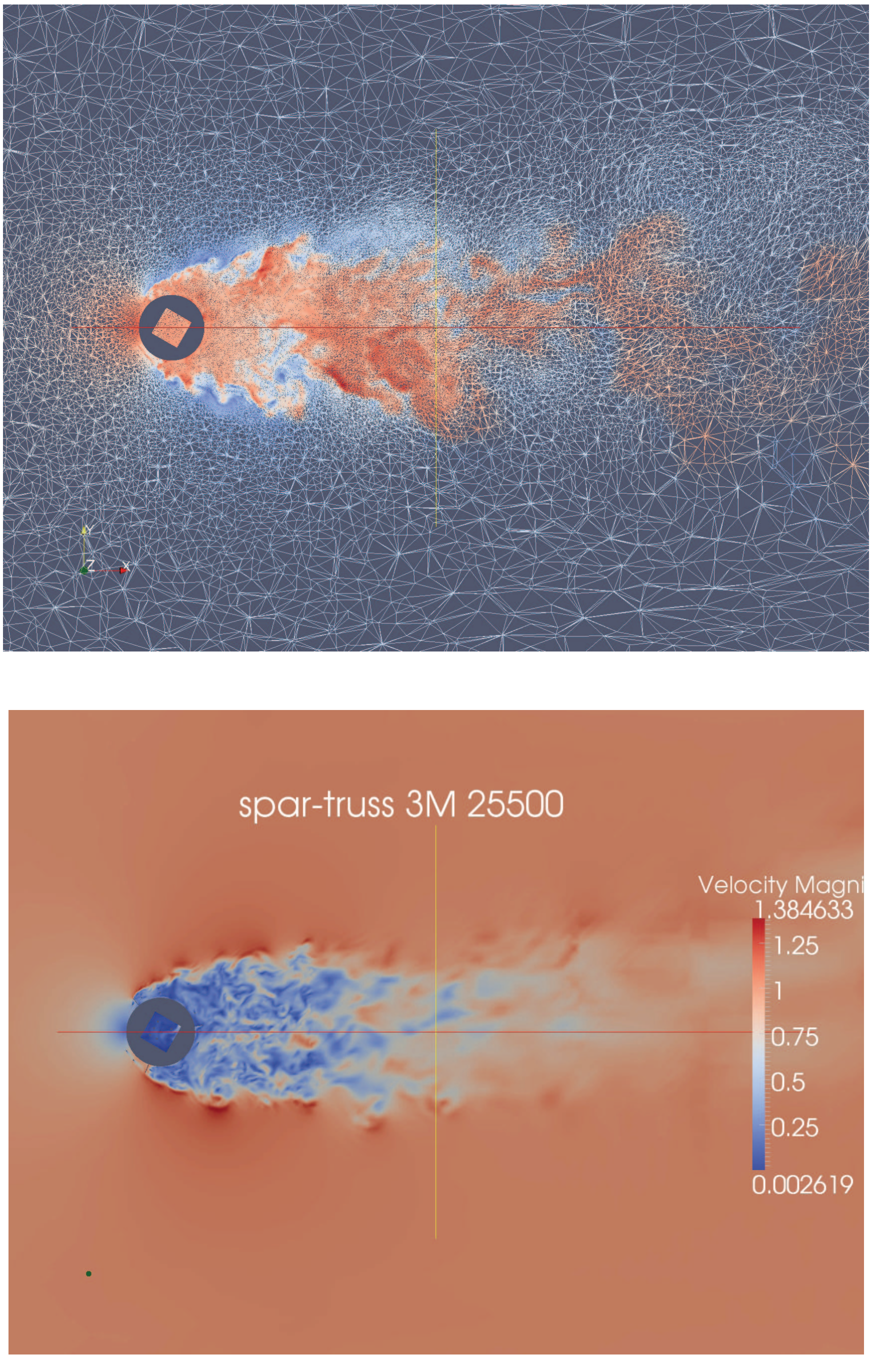

Figure 2: Mesh adaptive flow around an offshore platform: velocity module at two different times, the first one with mesh wireframe representation.