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ADAPTIVE REDUCED ORDER MULTISCALE FINITE ELEMENT METHODS BASED ON COMPONENT MODE SYNTHESIS

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Abstract. We present a reduced order finite element method based on the variational multiscale method together with a component mode synthesis representation for the fine scale part of the solution. We present an a posteriori error estimate in the energy norm for the discrete error in the approximation which measures the error associated with model reduction in the fine scale.

1 INTRODUCTION

In this contribution we briefly describe a recent multiscale finite element method, introduced in [6], which builds on using a reduced order model for the fine scale in a variational multiscale method, see [2] and the later developments [5].

Model reduction methods are commonly used to decrease the computational cost associated with simulations involving repeated use of large scale finite element models of for instance a complicated structure. The objective of model reduction methods is to find a low dimensional subspace of the finite element function space that still captures the essential behavior of the solution sufficiently well. A classical model reduction method is component mode synthesis (CMS), see [3].

In CMS the computational domain is split into subdomains and a reduced basis associated with the subdomain is constructed by solving localized constrained eigenvalue problems associated with the subdomains together with modes that represent the displacements of the interface between the subdomains, as in the Craig-Bampton method [1].

Here we construct a multiscale finite element method where the coarse scale is represented by piecewise linear continuous elements on a coarse mesh and the fine scale is defined by a CMS related approach on a refined mesh, using the coarse mesh elements as subdomains in the CMS method. The coupling modes are computed for each pair of neighboring elements and couple the response in the subdomains. Thus the fine scale is finally represented as a direct sum of functions with support in each element and functions associated with each edge supported in the two elements neighboring the edge. Adaptive reduction is accomplished by choosing a basis in each such subspace consisting of a truncated sequence of eigenmodes. The eigenmodes are numerically computed and capture fine scale effects.

We note that in the original CMS method the interface problem is global, which is a serious limitation since the reduced mass matrix is dense. In the multiscale method we present here we instead get a mass matrix with a block structure that is similar to finite element methods based on higher order polynomials. Furthermore, the size of all eigenvalue problems solved in the fine scale computations can be controlled by refining the coarse scale mesh.

We derive an a posteriori error estimate for the multiscale finite element method that can be used to automatically tune the number of subscale modes in an adaptive algorithm. For further details we refer to [6] and the previous work on a posteriori error estimates for component mode synthesis [4].

2 LINEAR ELASTICITY

Strong form: The equations of linear elasticity take the form: find displacements $\boldsymbol{u}$ such that

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) + \tau \boldsymbol{u} &= \boldsymbol{f}, & \boldsymbol{x} &\in \Omega, & \text{(1a)}\\
\boldsymbol{\sigma}(\boldsymbol{u}) &= 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I}, & \boldsymbol{x} &\in \Omega, & \text{(1b)}\\
\boldsymbol{u} &= \boldsymbol{0}, & \boldsymbol{x} &\in \Gamma_D, & \text{(1c)}\\
\boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{g}_N, & \boldsymbol{x} &\in \Gamma_N, & \text{(1d)}
\end{aligned}
$$

where $\tau \geq 0$ is a real parameter, $\boldsymbol{f}$ is a body force, $\boldsymbol{g}_N$ is a traction force, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the linear strain tensor, $\boldsymbol{\sigma}$ the stress tensor, $\boldsymbol{I}$ is the $d \times d$ identity matrix, and λ and μ are the Lamé parameters given by $\lambda = E\nu[(1+\nu)(1-2\nu)]^{-1}$ and $\mu = E[2(1+\nu)]^{-1}$, where E and ν is Young's modulus and Poisson's ratio respectively. The coefficients can have multiscale behavior, i.e. exhibit variation on a very fine scale or on multiple scales.

Weak form: The corresponding variational form of (1) reads: find $\boldsymbol{u} \in V = \{\boldsymbol{v} \in [H^1(\Omega)]^d : \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0}\}$ such that

$$
A(\boldsymbol{u}, \boldsymbol{v}) = b(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V, \tag{2}
$$

where $A(\cdot, \cdot)$ is the bilinear form

$$
A(\boldsymbol{v}, \boldsymbol{w}) = a(\boldsymbol{v}, \boldsymbol{w}) + \tau(\boldsymbol{v}, \boldsymbol{w}) \tag{3}
$$

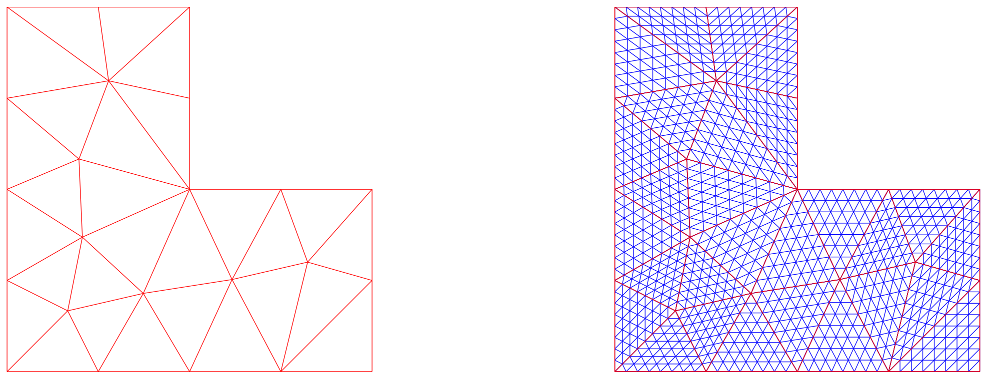

Figure 1: Left: coarse mesh $\mathcal{T}^H$ on an $L-$shaped domain. Left: fine mesh $\mathcal{T}^h$ obtained by a sequence of uniform refinements.

with

$$a(\boldsymbol{v}, \boldsymbol{v}) = 2(\mu\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{w})) + (\kappa\nabla\cdot\boldsymbol{v}, \nabla\cdot\boldsymbol{w}), \tag{4}$$

and $b(\cdot)$ is the linear form

$$b(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{g}_N, \boldsymbol{v})_{\Gamma_N}. \tag{5}$$

3 MULTISCALE METHOD

The mesh and finite element spaces: Let $\mathcal{T}^H$ be a coarse mesh on Ω consisting of shape regular triangles ($d = 2$) or tetrahedra ($d = 3$) and let $\mathcal{T}^h$ be a fine mesh obtained by a sequence of uniform refinements of $\mathcal{T}^H$. See Figure 3. Let $V^H \subset V^h$ be the corresponding spaces of continuous piecewise linear functions.

We then have the following splitting

$$V^h = V^H \oplus \left(\bigoplus_{E\in\mathcal{E}^H} V_E^h \right) \oplus \left(\bigoplus_{T\in\mathcal{T}^H} V_T^h \right) \tag{6}$$

Here $V_T^h \subset V^h$ is the space of functions with support in element $T \in \mathcal{T}^H$, $\mathcal{E}^H$ is the set of edges in the coarse mesh $\mathcal{T}^H$, and if the edge E is shared by elements T_1 and T_2 in $\mathcal{T}^H$ then the edge space V_E^h is defined by

$$V_E^h = \{v \in V^h \,:\, \mathrm{supp}(v) \subset T_1 \cup T_2, a(v, w) = 0 \;\forall w \in V_{T_1}^h \oplus V_{T_2}^h\} \tag{7}$$

Note that this means that the functions in V_E^h are uniquely determined, through harmonic extension, by the restriction to the edge.

Multiscale finite element space: To construct a multiscale basis in this finite element space we use Fourier expansions in terms of eigenmodes determined by the following eigenvalue problems. Reduction, is then obtained by truncating the Fourier expansion.

- Basis in V_T^h: Let $(\boldsymbol{Z}_i, \Lambda_i) \in V_T^h \times \mathbb{R}^+$, for $i = 1, 2, \ldots, \dim(V_T^h)$, be the eigenpairs defined by

  $$a(\boldsymbol{Z}, \boldsymbol{v}) = \Lambda(\boldsymbol{Z}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_T^h \tag{8}$$

  Using modal truncation we obtain a reduced subspace $V_T^{h,m_T} \subset V_T^h$, defined by

  $$V_T^{h,m_T} = \operatorname{span}\{\boldsymbol{Z}_i\}_{i=1}^{m_T}, \tag{9}$$

  where $m_T \ll \dim(V_T^h)$.

- Basis in V_E^h: Let $(\boldsymbol{Z}_i, \Lambda_i) \in V_E^h \times \mathbb{R}^+$, for $i = 1, 2, \ldots, \dim(V_E^h)$, be the eigenpairs defined by

  $$a(\boldsymbol{Z}, \boldsymbol{v}) = \Lambda(\boldsymbol{Z}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_E^h \tag{10}$$

  Using modal truncation we obtain a reduced subspace $V_E^{h,m_E} \subset V_E^h$, defined by

  $$V_E^{h,m_E} = \operatorname{span}\{\boldsymbol{Z}_i\}_{i=1}^{m_E}, \tag{11}$$

  where $m_E \ll \dim(V_E^h)$.

Finally, we arrive at the following reduced order multiscale finite element space

$$V^{h,\boldsymbol{m}} = V^H \oplus \left(\bigoplus_{E \in \mathcal{E}^H} V_E^{h,m_E} \right) \oplus \left(\bigoplus_{T \in \mathcal{T}^H} V_T^{h,m_T} \right) \tag{12}$$

where $\boldsymbol{m} = (\cup_{E \in \mathcal{E}^H} m_E) \cup (\cup_{T \in \mathcal{T}^H} m_T)$ is the multi index containing the indices m_E and m_T for all edges $E \in \mathcal{E}^H$ and elements $T \in \mathcal{T}^H$.

Multiscale finite element method: The multiscale method is then directly obtained by using this reduced order space in the standard variational formulation: find $\boldsymbol{U}^{\boldsymbol{m}} \in V^{h,\boldsymbol{m}}$ such that

$$A(\boldsymbol{U}^{\boldsymbol{m}}, \boldsymbol{v}) = b(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V^{h,\boldsymbol{m}}, \tag{13}$$

Note that this is a coupled system involving both the coarse piecewise linear functions and the edge and element spaces spanned by the eigenmodes define above. Fine scale effects are captured in computations of the eigenfunctions on the fine mesh.

4 A POSTERIORI ERROR ESTIMATE

A posteriori error estimates: Let $|||\cdot|||$ denote the energy norm, $|||v|||^2 = A(v,v)$ and let $\boldsymbol{U}^h$ denote the standard finite element solution in V^h. Then we have the following a posteriori error estimate

$$|||\boldsymbol{U}^h - \boldsymbol{U}^{\boldsymbol{m}}|||^2 \leq \sum_{E\in\mathcal{E}^H} \eta_E^2 + \sum_{E\in\mathcal{T}^H} \eta_T^2. \tag{14}$$

Here we introduced the following subspace indicators

$$\eta_E^2 = \frac{\|\boldsymbol{R}_E(\boldsymbol{U}^{\boldsymbol{m}})\|^2}{\Lambda_{E,m_E+1}}, \quad E \in \mathcal{E}^H, \tag{15}$$

$$\eta_T^2 = \frac{\|\boldsymbol{R}_T(\boldsymbol{U}^{\boldsymbol{m}})\|^2}{\Lambda_{T,m_T+1}}, \quad T \in \mathcal{T}^H, \tag{16}$$

where the subspace residual $\boldsymbol{R}_I : V_I^h \to V_I^h$, is defined by

$$(\boldsymbol{R}_I(\boldsymbol{w}), \boldsymbol{v}) = b(\boldsymbol{v}) - A(\boldsymbol{w}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_I^h, \quad I \in \mathcal{E}^H \cup \mathcal{T}^H \tag{17}$$

The indicators measure the error contribution due to reduction in the corresponding subspaces V_E^{h,m_E}, $E \in \mathcal{E}$, and V_T^{h,m_T}, $T \in \mathcal{T}$.

Adaptive algorithm: Based on the a posteriori error estimate (14) we may construct an adaptive solution procedure as follows:

1. Start with a guess of the subspace dimensions in V_E^{h,m_E} and V_T^{h,m_T}.
2. Solve the problem (13) and compute the subspace indicators (15) and (16).
3. If an indicator is large according to some refinement criterion, increase the number of modes in that subspace.
4. If $\sum_{E\in\mathcal{E}} \eta_E^2 + \sum_{T\in\mathcal{T}} \eta_T^2 <$ TOL, where TOL is a predetermined tolerance, stop. Otherwise, go to 2.

5 NUMERICAL EXAMPLE

We finally consider linear elasticity with $\tau = 0$, Young's modulus $E = 1$, and Poisson's ratio $\nu = 0.3$ on the $L-$shaped domain clamped, and free on the reminder of the boundary, at one side and exposed to a gravity force F_g acting on the whole domain, see Figure 5 (left). As is well known the solution is singular in the corner. We use an unstructured triangulation to construct the coarse mesh and a sequence of uniform refinements to construct the fine scale mesh, see 3. In figure 5 (right) we compare the adaptive

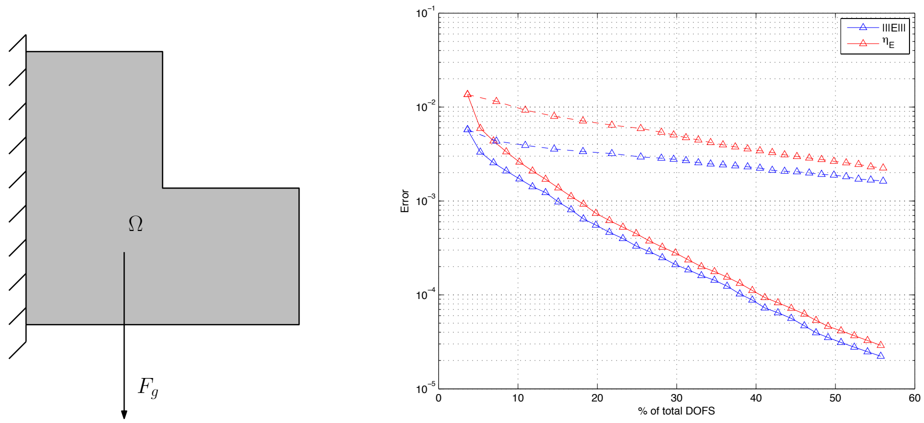

Figure 2: Left: set up of the problem in the numerical example. Right: decay of the error for the reduced displacement $\boldsymbol{U}_r$ using a uniform refinement strategy (dashed line) compared to an adaptive strategy (solid line).

strategy described above with a uniform strategy. We plot the estimated energy norm error compared to the actual energy norm error. We note that the error estimate is sharp and that the adaptive strategy outperforms the uniform strategy. We also note that the adaptive method actually produces an exponentially convergent sequence of approximate solutions.

REFERENCES

[1] Craig, R.R. and Bampton, M.C.C., Coupling of substructures for dynamic analysis. *AIAA Journal.* (1968) **6**:1313–1321

[2] Hughes, T.J.R., Feijóo, G.R., Mazzei, L., and Quincy, J.B. The variational multiscale methoda paradigm for computational mechanics. *Computer methods in applied mechanics and engineering.* (1998) **166.1**:1313-1321.

[3] Hurty, W.C. Dynamic analysis of structural systems using component modes, *AIAA Journal.* (1965) **4**:678–685

[4] Jakobsson, H., Bengzon, F., and Larson, M.G. Adaptive component mode synthesis in Linear Elasticity. *Int. J. Numer. Meth. Engng.* (2011) **200.41–44**:829–844.

[5] Larson, M.G. and Målqvist, A. Adaptive variational multiscale methods based on a posteriori error estimation: energy norm estimates for elliptic problems. *Computer methods in applied mechanics and engineering.* (2007) **196.21**:2313–2324

[6] Troeng, T., Jakobsson, H., and Larson, M.G. A reduced order multiscale finite element method based on component mode synthesis. (2013) Technical report, Umeå University.