Hybridizable discontinuous Galerkin (HDG) method for Oseen flow

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1 Introduction

This short note complements the computer lab on hybridizable discontinuous Galerkin for the approximation of the Oseen flow and is the natural follow-up of the previous sessions. The associated academic Matlab code solves problems in two dimensions using meshes based on triangular elements. This lab session is devoted to the experimental analysis of the role of the stabilization parameter in problems featuring advection phenomena. For a general analysis of HDG for Oseen flow, the interested reader is referred to [1].

2 Problem statement

Let $\Omega \subset \mathbb{R}^{\mathtt{n_{sd}}}$ be an open bounded domain with boundary $\partial\Omega$. The strong form of the steady Oseen equation with non-homogeneous Dirichlet boundary conditions reads as follows:

$$\begin{cases} -\boldsymbol{\nabla}\cdot(\mathbf{K}\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u}\otimes\boldsymbol{a} - p\mathbf{I}_{\mathtt{n_{sd}}}) = \boldsymbol{s} & \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u}_D & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the couple $(\boldsymbol{u}, p)$ represents the velocity and pressure fields associated with the problem, $\mathbf{K} = \nu\mathbf{I}_{\mathtt{n_{sd}}}$, $\nu > 0$ is the viscosity matrix, $\boldsymbol{a}$ a divergence-free advection field and $\boldsymbol{s}$ and $\boldsymbol{u}_D$ respectively are the volumetric source term and the Dirichlet boundary datum to impose the value of the velocity on $\partial\Omega$. Remark that owing to the purely Dirichlet boundary condition, the pressure in (1) is determined up to a constant. Hence, the following additional constraint enforcing zero mean value of the pressure field is introduced:

$$\int_{\partial\Omega} p \; d\Gamma = 0. \tag{2}$$

Assume that Ω is partitioned in $\mathtt{n_{el}}$ disjoint subdomains Ω_i's

$$\overline{\Omega} = \bigcup_{i=1}^{\mathtt{n_{el}}} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j, \quad \Omega^h := \bigcup_{i=1}^{\mathtt{n_{el}}} \Omega_i,$$

whose boundaries $\partial\Omega_i$ define the internal interface Γ

$$\Gamma := \left[\bigcup_{i=1}^{\mathtt{n_{el}}} \partial\Omega_i\right] \setminus \partial\Omega. \tag{3}$$

Moreover, the *jump* operator $[\![\cdot]\!]$ along each portion of the interface Γ is defined as the sum of the values of the quantity under analysis in the elements Ω_i and Ω_j respectively on the left and right sides of the interface (cf. [2]), that is ,

$$[\![\odot]\!] = \odot_i + \odot_j.$$

It is important to observe that this definition always requires the normal vector $\boldsymbol{n}$ in the argument and produces functions in the same space as the argument.

The original strong problem (1) can thus be rewritten equivalently in mixed form on the broken domain as follows:

$$\left\{\begin{aligned} \mathbf{L} - \boldsymbol{\nabla}\boldsymbol{u} &= \mathbf{0} && \text{in } \Omega_i \text{, and for } i = 1, \dots, \mathtt{n_{el}}, \\ -\boldsymbol{\nabla} \cdot \big(\mathbf{K}\,\mathbf{L} - \boldsymbol{u} \otimes \boldsymbol{a} - p\mathbf{I}_{\mathtt{n_{sd}}}\big) &= \boldsymbol{s} && \text{in } \Omega_i \text{, and for } i = 1, \dots, \mathtt{n_{el}}, \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 && \text{in } \Omega_i \text{, and for } i = 1, \dots, \mathtt{n_{el}}, \\ \boldsymbol{u} &= \boldsymbol{u}_D && \text{on } \partial\Omega, \\ [\![\boldsymbol{u} \otimes \boldsymbol{n}]\!] &= \mathbf{0} && \text{on } \Gamma, \\ [\![-(\mathbf{K}\,\mathbf{L})\boldsymbol{n} + (\boldsymbol{u} \otimes \boldsymbol{a})\boldsymbol{n} + p\boldsymbol{n}]\!] &= \mathbf{0} && \text{on } \Gamma. \end{aligned}\right. \tag{4}$$

where last two equations - also known as *transmission conditions* - enforce the continuity of respectively the primal variable and the normal trace of the flux across the interface Γ. In order for pressure to be uniquely defined, (4) is coupled again with the constraint (2).

3 The hybridizable discontinuous Galerkin (HDG) formulation

The hybridizable discontinuous Galerkin method was first introduced in [3]. In this section, the method is used to approximate the Oseen equation [1]. First, following the notation in [4], the discrete functional spaces are introduced:

$$\begin{aligned} \mathcal{V}^h(\Omega) &:= \{v \in \mathcal{L}_2(\Omega) : v|_{\Omega_i} \in \mathcal{P}^{k_i}(\Omega_i)\ \forall \Omega_i\,,\ i = 1, \dots, \mathtt{n_{el}}\}, \\ \mathcal{M}^h(S) &:= \{\hat{v} \in \mathcal{L}_2(S) : \hat{v}|_{\Gamma_i} \in \mathcal{P}^{k_i}(\Gamma_i)\ \forall \Gamma_i \subset S \subseteq \Gamma \cup \partial\Omega\}, \end{aligned}$$

where $\mathcal{P}^{k_i}(\Omega_i)$ and $\mathcal{P}^{k_i}(\Gamma_i)$ stand for the spaces of polynomial functions of complete degree at most k_i respectively in Ω_i and on Γ_i. Moreover, recall the notation $(p, q)_V := \int_V pq\ d\Omega$ and $\langle p, q\rangle_S := \int_S pq\ d\Gamma$ for the classical internal products in $\mathcal{L}_2(V)$, $V \subseteq \Omega$ and $\mathcal{L}_2(S)$, $S \subseteq \Gamma$.

The hybridizable discontinuous Galerkin formulation of the Oseen equation introduced in section 2 is composed by $\mathtt{n_{el}}$ local problems defined on the interior of the elements Ω_i's and a global problem set on the internal skeleton Γ. In the following subsections, the aforementioned problems will be detailed.

3.1 HDG local problems

The local problems determine $(\boldsymbol{u}_i, \mathbf{L}_i, p_i)$ for each element Ω_i, $i = 1, \dots \mathtt{n_{el}}$ as functions of a new variable $\hat{\boldsymbol{u}}$ defined along the interface Γ and acting as a Dirichlet boundary condition, namely

$$
\begin{cases}
\mathbf{L}_i - \boldsymbol{\nabla}\boldsymbol{u}_i = \mathbf{0} & \text{in } \Omega_i, \\
-\boldsymbol{\nabla}\cdot\big(\mathbf{K}\,\mathbf{L}_i - \boldsymbol{u}_i\otimes\boldsymbol{a} - p_i\mathbf{I}_{\mathtt{n_{sd}}}\big) = \boldsymbol{s} & \text{in } \Omega_i, \\
\boldsymbol{\nabla}\cdot\boldsymbol{u}_i = 0 & \text{in } \Omega_i, \\
\boldsymbol{u}_i = \boldsymbol{u}_D & \text{on } \partial\Omega_i\cap\partial\Omega, \\
\boldsymbol{u}_i = \hat{\boldsymbol{u}} & \text{on } \partial\Omega_i\setminus\partial\Omega,
\end{cases}
\tag{5}
$$

Henceforth, assume that the hybrid variable $\hat{\boldsymbol{u}}$ is defined on both the internal skeleton Γ and the external boundary $\partial\Omega$. More precisely, let $\hat{\boldsymbol{u}} = \boldsymbol{u}_D$ on $\partial\Omega$. It follows that the boundary conditions in (5) may be rewritten simply as

$$
\boldsymbol{u}_i = \hat{\boldsymbol{u}} \text{ on } \partial\Omega_i. \tag{6}
$$

By multiplying the first three equations in (5) by test functions belonging to appropriate functional spaces and integrating by parts, the following weak form of the local problems is derived. For $i = 1,\dots\mathtt{n_{el}}$, seek $(\boldsymbol{u}_i^h, \mathbf{L}_i^h, p_i^h) \in [\mathcal{V}^h(\Omega_i)]^{\mathtt{n_{sd}}} \times [\mathcal{V}^h(\Omega_i)]^{\mathtt{n_{sd}}\times\mathtt{n_{sd}}} \times \mathcal{V}^h(\Omega_i)$ such that for all $(\boldsymbol{w}, \mathbf{G}, q) \in [\mathcal{V}^h(\Omega_i)]^{\mathtt{n_{sd}}} \times [\mathcal{V}^h(\Omega_i)]^{\mathtt{n_{sd}}\times\mathtt{n_{sd}}} \times \mathcal{V}^h(\Omega_i)$ it holds

$$
(\mathbf{G}, \mathbf{L}_i^h)_{\Omega_i} + (\boldsymbol{\nabla}\cdot\mathbf{G}, \boldsymbol{u}_i^h)_{\Omega_i} = \langle \mathbf{G}\boldsymbol{n}_i, \hat{\boldsymbol{u}}^h\rangle_{\partial\Omega_i}, \tag{7a}
$$

$$
\begin{aligned}
-(\boldsymbol{w}, \boldsymbol{\nabla}\cdot(\mathbf{K}\,\mathbf{L}_i^h))_{\Omega_i} + (\boldsymbol{w}, \mathbf{L}_i^h\,\boldsymbol{a})_{\Omega_i} &+ (\boldsymbol{w}, \boldsymbol{\nabla}p_i^h)_{\Omega_i} \\
&+ \langle \boldsymbol{w}, \tau_i\boldsymbol{u}_i^h\rangle_{\partial\Omega_i} = (\boldsymbol{w}, \boldsymbol{s})_{\Omega_i} + \langle \boldsymbol{w}, \tau_i\hat{\boldsymbol{u}}^h\rangle_{\partial\Omega_i},
\end{aligned}
\tag{7b}
$$

$$
(\boldsymbol{\nabla}q, \boldsymbol{u}_i^h)_{\Omega_i} = \langle q, \hat{\boldsymbol{u}}^h\cdot\boldsymbol{n}_i\rangle_{\partial\Omega_i}, \tag{7c}
$$

$$
\langle 1, p_i^h\rangle_{\partial\Omega_i} = \rho_i, \tag{7d}
$$

where the trace of the numerical flux is defined element-by-element as

$$
\big(\widehat{-\mathbf{K}\,\mathbf{L}_i^h + \boldsymbol{u}_i^h\otimes\boldsymbol{a} + p_i^h\mathbf{I}_{\mathtt{n_{sd}}}}\big)\boldsymbol{n}_i = -\big(\mathbf{K}\,\mathbf{L}_i^h\big)\boldsymbol{n}_i + \big(\boldsymbol{u}_i^h\otimes\boldsymbol{a}\big)\boldsymbol{n}_i + p_i^h\boldsymbol{n}_i + \tau_i(\boldsymbol{u}_i^h - \hat{\boldsymbol{u}}^h). \tag{8}
$$

Remark that the momentum equation (7b) has been integrated twice in order to make the flux associated with the operator (first three terms on the right-hand side of (8)) vanish by leaving solely the stabilization term $\langle \boldsymbol{w}, \tau_i(\boldsymbol{u}_i^h - \hat{\boldsymbol{u}}^h)\rangle_{\partial\Omega_i}$. Moreover, in order to account the purely Dirichlet boundary conditions (6) applied on the local problem, the constraint (7d) has been introduced.

The stabilization parameter τ_i plays a crucial role in the accuracy and convergence of the HDG method (cf. e.g. [1, 5, 6]) and may assume different values on each face of the boundary $\partial\Omega_i$. Its role in the HDG approximation of the Oseen flow is the main subject of this lab session and will be investigated via experimental analysis.

3.2 HDG global problem

As previously mentioned, the hybrid variable $\hat{\boldsymbol{u}}^h$ introduced in (7) is the unknown of a global problem which accounts for the transmission conditions in (4): seek $\hat{\boldsymbol{u}}^h \in [\mathcal{M}^h(\Gamma\cup$

$\partial\Omega)]^{\mathtt{n_{sd}}}$ such that $\hat{\boldsymbol{u}}^h = \boldsymbol{u}_D$ on $\partial\Omega$ and for all $\hat{\boldsymbol{v}} \in [\mathcal{M}^h(\Gamma \cup \partial\Omega)]^{\mathtt{n_{sd}}}$ it holds

$$
\begin{aligned}
\sum_{i=1}^{\mathtt{n_{el}}} \Big\{ -\langle \hat{\boldsymbol{v}}, \big(\mathbf{K}\,\mathbf{L}_i^h\big)\boldsymbol{n}_i \rangle_{\partial\Omega_i \setminus \partial\Omega} + \langle \hat{\boldsymbol{v}}, \big(\hat{\boldsymbol{u}}^h \otimes \boldsymbol{a}\big)\boldsymbol{n}_i \rangle_{\partial\Omega_i \cap \Gamma} + \langle \hat{\boldsymbol{v}}, p_i^h \boldsymbol{n}_i \rangle_{\partial\Omega_i \setminus \partial\Omega} & \\
+ \langle \hat{\boldsymbol{v}}, \tau_i\, \boldsymbol{u}_i^h \rangle_{\partial\Omega_i \setminus \partial\Omega} - \langle \hat{\boldsymbol{v}}, \tau_i\, \hat{\boldsymbol{u}}^h \rangle_{\partial\Omega_i \setminus \partial\Omega} \Big\} &= 0
\end{aligned}
, \tag{9a}
$$

$$
\langle 1, \hat{\boldsymbol{u}}^h \cdot \boldsymbol{n}_i \rangle_{\partial\Omega_i} = 0 \qquad \text{for } i = 1, \dots, \mathtt{n_{el}} \tag{9b}
$$

The local problems (7) featuring purely Dirichlet boundary conditions, pressure is known up to a constant. In order to impose an additional constraint, equation (7d) sets the mean value of p_i^h on the boundary of all elements Ω_i's, for $i = 1, \dots, \mathtt{n_{el}}$. Thus, the compatibility condition (9b) to weakly enforce a divergence-free velocity field on each element is introduced.

3.3 Local post-process of the velocity field

By solving an additional problem element-by-element, a post-processed velocity field may be computed. More precisely, when using an approximation based on polynomials of order k, the optimal order of convergence $k+1$ of the gradient of the velocity $\mathbf{L}$ is exploited to obtain a velocity field $\boldsymbol{u}^*$ superconvenverging with order $k+2$. The idea of post-processing the velocity field was first proposed in [7] and stems from the BDM-projection used in mixed methods [8, 9] to obtain optimal order of convergence for the flux. The post-processed velocity is obtained by solving the following problem in each element:

$$
\begin{cases}
\boldsymbol{\nabla} \cdot \big(\boldsymbol{\nabla} \boldsymbol{u}_i^*\big) = \boldsymbol{\nabla} \cdot \mathbf{L}_i^h & \text{in } \Omega_i, \\
\big(\boldsymbol{\nabla} \boldsymbol{u}_i^*\big)\boldsymbol{n}_i = \mathbf{L}_i^h \boldsymbol{n}_i & \text{on } \partial\Omega_i,
\end{cases} \tag{10}
$$

with the additional constraint

$$
\int_{\Omega_i} \boldsymbol{u}_i^* \; d\Omega = \int_{\Omega_i} \boldsymbol{u}_i^h \; d\Omega \quad \text{for } i = 1, \dots, \mathtt{n_{el}}. \tag{11}
$$

Henceforth, to simplify the notation the superindex ^{h} expressing the discrete approximations and the subindex $_i$ indicating the element will be dropped, unless needed in order to follow the development.

3.4 Assembly of the matrices

Consider an element-by-element nodal interpolation for the spatial unknown functions defined as

$$
\hat{\boldsymbol{u}}(\boldsymbol{x}) \approx \hat{\boldsymbol{u}}^h(\boldsymbol{x}) = \sum_{j=1}^{\mathtt{n_{fn}}} \widehat{N}_j(\boldsymbol{x})\, \widehat{\mathbf{u}}_j \qquad \in [\mathcal{M}^h(\Gamma \cup \partial\Omega)]^{\mathtt{n_{sd}}}, \tag{12a}
$$

$$\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}^h(\boldsymbol{x}) = \sum_{j=1}^{\mathtt{n_{en}}} N_j(\boldsymbol{x})\, \mathbf{u}_j \qquad \in [\mathcal{V}^h(\Omega^h)]^{\mathtt{n_{sd}}}, \tag{12b}$$

$$\mathbf{L}(\boldsymbol{x}) \approx \mathbf{L}^h(\boldsymbol{x}) = \sum_{j=1}^{\mathtt{n_{en}}} N_j(\boldsymbol{x})\, \mathbf{L}_j \qquad \in [\mathcal{V}^h(\Omega^h)]^{\mathtt{n_{sd}}\times\mathtt{n_{sd}}}, \tag{12c}$$

$$p(\boldsymbol{x}) \approx p^h(\boldsymbol{x}) = \sum_{j=1}^{\mathtt{n_{en}}} N_j(\boldsymbol{x})\, p_j \qquad \in \mathcal{V}^h(\Omega^h), \tag{12d}$$

$$\tag{12e}$$

where $\widehat{\mathbf{u}}_j$, $\mathbf{u}_j$, $\mathbf{L}_j$ and p_j are nodal values, N_j are polynomial shape functions of order k in each element, $\mathtt{n_{en}}$ is the number of nodes per element, $\widehat{N}_j$ are the polynomial shape functions of order k on each element face/edge, and $\mathtt{n_{fn}}$ is the corresponding number of nodes per face/edge. Hence, the vectors $\mathbf{u}$, $\mathbf{L}$ and $\mathbf{p}$ are defined for each element $i = 1, \ldots, \mathtt{n_{el}}$ and are respectively of dimension $\mathtt{n_{en}}\mathtt{n_{sd}}$, $\mathtt{n_{en}}\mathtt{n_{sd}}^2$ and $\mathtt{n_{en}}$. The vector $\widehat{\mathbf{u}}$ is defined globally over the skeleton of the mesh and its dimension corresponds to the number of nodes on $\Gamma \cup \partial\Omega$. More precisely,

$$\dim(\widehat{\mathbf{u}}) = \sum_{k=1}^{\mathtt{n_{ef}}} \mathtt{n}_{\mathtt{fn}}^k,$$

where $\mathtt{n_{ef}}$ is the number of element faces/edges of the mesh skeleton $\Gamma \cup \partial\Omega$ and $\mathtt{n}_{\mathtt{fn}}^k$ is the number of nodes on the k-th face. Remark that owing to the boundary condition $\hat{\boldsymbol{u}}^h = \boldsymbol{u}_D$ on $\partial\Omega$, solely the nodes belonging to Γ are unknowns of the previously introduced global problem.

Algebraic system arising from the local problem

The matrix formulation of the local problem (7) for each element Ω_i for $i = 1, \ldots, \mathtt{n_{el}}$ reads as follows:

$$\begin{bmatrix} \mathbf{A}_{uu} & \mathbf{A}_{uL} & \mathbf{A}_{up} & \mathbf{0} \\ \mathbf{A}_{Lu} & \mathbf{A}_{LL} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{pu} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\rho p}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{\rho p} & \mathbf{0} \end{bmatrix}_i \begin{Bmatrix} \mathbf{u} \\ \mathbf{L} \\ \mathbf{p} \\ \lambda \end{Bmatrix}_i = \begin{Bmatrix} \mathbf{f}_u \\ \mathbf{f}_L \\ \mathbf{f}_p \\ 0 \end{Bmatrix}_i + \begin{bmatrix} \mathbf{A}_{u\hat{u}} \\ \mathbf{A}_{L\hat{u}} \\ \mathbf{A}_{p\hat{u}} \\ 0 \end{bmatrix}_i \widehat{\mathbf{u}}_i + \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{Bmatrix}_i \rho_i \tag{13}$$

where a Lagrange multiplier λ has been introduced to handle the additional constraint to enforce uniqueness of the pressure field and the matrix $\mathbf{A}_{\rho p}$ arises from the left-hand side of equation (7d). Remark that the variable ρ_i features solely one degree of freedom per element.

The local block matrices $A_\odot$ are computed exploiting the interpolation framework discussed above and the corresponding forms $a_\odot$ in appendix A.

Algebraic system arising from the global problem

Similarly, the following algebraic formulation may be derived for the global problem (9):

$$\sum_{i=1}^{\mathtt{n_{el}}} \left\{ \begin{bmatrix} \mathbf{A}_{\hat{u}u} & \mathbf{A}_{\hat{u}L} & \mathbf{A}_{\hat{u}p} \end{bmatrix}_i \begin{Bmatrix} \mathbf{u} \\ \mathbf{L} \\ \mathbf{p} \end{Bmatrix}_i + \begin{bmatrix} \mathbf{A}_{\hat{u}\hat{u}} \end{bmatrix}_i \widehat{\mathbf{u}}_i \right\} = 0 \tag{14}$$

where the block matrices are computed as mentioned above starting from the forms discussed in appendix A.

By computing the inverse of the matrix in (13), the solution $\{\mathbf{u}, \mathbf{L}, \mathbf{p}, \lambda\}_i^T$ may be written as a function of $\widehat{\mathbf{u}}_i$. By introducing a restriction operator $\mathcal{R}$ that neglects the last line associated with the Lagrange multiplier and by plugging the resulting vector $\{\mathbf{u}, \mathbf{L}, \mathbf{p}\}_i^T$ into (14), the following global problem is derived

$$\widehat{\mathbf{A}}_{\hat{u}\hat{u}}\widehat{\mathbf{u}} + \widehat{\mathbf{A}}_{\hat{u}\rho}\rho = \widehat{\mathbf{f}}_{\hat{u}} \tag{15}$$

where

$$\widehat{\mathbf{A}}_{\hat{u}\hat{u}} = \mathsf{A}_{i=1}^{\mathtt{n_{el}}} \begin{bmatrix} \mathbf{A}_{\hat{u}u} & \mathbf{A}_{\hat{u}L} & \mathbf{A}_{\hat{u}p} \end{bmatrix}_i \mathcal{R} \left(\begin{bmatrix} \mathbf{A}_{uu} & \mathbf{A}_{uL} & \mathbf{A}_{up} & \mathbf{0} \\ \mathbf{A}_{Lu} & \mathbf{A}_{LL} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{pu} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\rho p}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{\rho p} & \mathbf{0} \end{bmatrix}_i^{-1} \begin{bmatrix} \mathbf{A}_{u\hat{u}} \\ \mathbf{A}_{L\hat{u}} \\ \mathbf{A}_{p\hat{u}} \\ 0 \end{bmatrix}_i \right) + \begin{bmatrix} \mathbf{A}_{\hat{u}\hat{u}} \end{bmatrix}_i,$$

$$\widehat{\mathbf{A}}_{\hat{u}\rho} = \mathsf{A}_{i=1}^{\mathtt{n_{el}}} \begin{bmatrix} \mathbf{A}_{\hat{u}u} & \mathbf{A}_{\hat{u}L} & \mathbf{A}_{\hat{u}p} \end{bmatrix}_i \mathcal{R} \left(\begin{bmatrix} \mathbf{A}_{uu} & \mathbf{A}_{uL} & \mathbf{A}_{up} & \mathbf{0} \\ \mathbf{A}_{Lu} & \mathbf{A}_{LL} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{pu} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\rho p}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{\rho p} & \mathbf{0} \end{bmatrix}_i^{-1} \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{Bmatrix}_i \right),$$

$$\widehat{\mathbf{f}}_{\hat{u}} = -\mathsf{A}_{i=1}^{\mathtt{n_{el}}} \begin{bmatrix} \mathbf{A}_{\hat{u}u} & \mathbf{A}_{\hat{u}L} & \mathbf{A}_{\hat{u}p} \end{bmatrix}_i \mathcal{R} \left(\begin{bmatrix} \mathbf{A}_{uu} & \mathbf{A}_{uL} & \mathbf{A}_{up} & \mathbf{0} \\ \mathbf{A}_{Lu} & \mathbf{A}_{LL} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{pu} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\rho p}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{\rho p} & \mathbf{0} \end{bmatrix}_i^{-1} \begin{Bmatrix} \mathbf{f}_u \\ \mathbf{f}_L \\ \mathbf{f}_p \\ 0 \end{Bmatrix}_i \right).$$

Accounting for the global purely Dirichlet boundary condition

The compatibility condition (9b) may be rewritten as

$$\widehat{\mathbf{A}}_{\rho\hat{u}}\widehat{\mathbf{u}} = 0, \tag{16}$$

where $\widehat{\mathbf{A}}_{\rho\hat{u}}$ is the matrix associated with the bilinear form on the left-hand side of (9b). The final algebraic global system is obtained from (15) and (16):

$$\begin{bmatrix} \widehat{\mathbf{A}}_{\hat{u}\hat{u}} & \widehat{\mathbf{A}}_{\hat{u}\rho} \\ \widehat{\mathbf{A}}_{\rho\hat{u}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \widehat{\mathbf{u}} \\ \rho \end{Bmatrix} = \begin{Bmatrix} \widehat{\mathbf{f}}_{\hat{u}} \\ 0 \end{Bmatrix}, \tag{17}$$

whose unknowns are the the trace of the velocity $\widehat{\mathbf{u}}$ defined on the mesh skeleton $\Gamma \cup \partial\Omega$ and the mean boundary pressures ρ defined in all the elements.

4 Exercise: the Kovasznay flow [oseen/ex_2]

Let $\Omega \subset \mathbb{R}^2$ be the square domain $[-1,1]\times[-1,1]$ and consider the analytical solution of the incompressible Navier-Stokes equations known as Kovasznay flow [10]:

$$\begin{aligned}
u_1 &= 1 - e^{\lambda(x+1)} \cos\Big(2\pi\Big(y+\frac{1}{2}\Big)\Big),\\
u_2 &= \frac{\lambda}{2\pi} e^{\lambda(x+1)} \sin\Big(2\pi\Big(y+\frac{1}{2}\Big)\Big),\\
p &= -\frac{1}{2} e^{2\lambda(x+1)} + C,
\end{aligned}$$

where $\lambda := 1/(2\nu) - \sqrt{1/(4\nu^2)+4\pi^2}$ and C is a constant such that the exact pressure has zero mean value on the boundary $\partial\Omega$. By setting the advection field $\boldsymbol{a}$ equal to the analytical solution $\boldsymbol{u}$ and by computing appropriately the Dirichlet boundary datum $\boldsymbol{u}_D$ and the source term $\boldsymbol{s}$, the Kovasznay flow is also a solution of the Oseen equation.

Diffusion-dominated and advection-dominated regimes Solve the Oseen equation for $\nu = 10^{-1}$ and $\nu = 10^{-4}$ by considering a stabilization term constant on each face Γ_i of the triangulation. More precisely, set $\tau_i = \tau_i^{(1)} := \nu/\ell$, ℓ being a characteristic length of the domain. Repeat the experiments by setting $\tau_i = 10^{\kappa}\nu/\ell$, with $\kappa = 1,\dots,3$.

Incorporating the advection information into the stabilization term Implement the following two expressions of the stabilization parameter constant on each face Γ_i:

$$\tau_i^{(2)} := \frac{\nu}{\ell} + \frac{1}{2}\max_{\boldsymbol{x}\in\Gamma_i}\big|\boldsymbol{a}(\boldsymbol{x})\cdot\boldsymbol{n}\big|, \quad \tau_i^{(3)} := \frac{\nu}{\ell} + \frac{1}{2}\max_{\boldsymbol{x}\in\Gamma_i}\big\{\boldsymbol{a}(\boldsymbol{x})\cdot\boldsymbol{n}, 0\big\}$$

Optimal order of convergence and superconvergence of the post-processed velocity Recall that for a polynomial approximation of order k, the optimal order of convergence for velocity is $k+1$ and the one for the post-processed velocity is $k+2$. Verify experimentally the order of convergence of the velocity for $\nu = 10^{-1}$ and $\nu = 10^{-4}$ for different meshes `mesh2_P`,…,`mesh5_P` and for different order of approximation `degree` $= 1,\dots,5$. Compare the results obtained using different expressions of the stabilization parameter.

A Bilinear and linear forms

Forms used in the HDG global problem (14):

$$\begin{aligned}
a_{\hat{u}u}(\hat{\boldsymbol{v}},\boldsymbol{u}) &:= \langle \hat{\boldsymbol{v}}, \tau_i \boldsymbol{u}_i\rangle_{\partial\Omega_i\setminus\partial\Omega}, &\quad a_{\hat{u}L}(\hat{\boldsymbol{v}},\mathbf{L}) &:= -\langle \hat{\boldsymbol{v}}, \big(\mathbf{K}\mathbf{L}_i\big)\boldsymbol{n}_i\rangle_{\partial\Omega_i\setminus\partial\Omega},\\
a_{\hat{u}p}(\hat{\boldsymbol{v}},p) &:= \langle \hat{\boldsymbol{v}}, p_i \boldsymbol{n}_i\rangle_{\partial\Omega_i\setminus\partial\Omega}, &\quad a_{\hat{u}\hat{u}}(\hat{\boldsymbol{v}},\hat{\boldsymbol{u}}) &:= \langle \hat{\boldsymbol{v}}, \big(\hat{\boldsymbol{u}}\otimes\boldsymbol{a}\big)\boldsymbol{n}_i\rangle_{\partial\Omega_i\cap\Gamma} - \langle \hat{\boldsymbol{v}}, \tau_i\hat{\boldsymbol{u}}\rangle_{\partial\Omega_i\setminus\partial\Omega}.
\end{aligned}$$

Forms used in the HDG local problem (13):

$$
\begin{aligned}
a_{uu}(\boldsymbol{w},\boldsymbol{u}) &:= \langle \boldsymbol{w}, \tau_i \boldsymbol{u}_i \rangle_{\partial\Omega_i}, & a_{uL}(\boldsymbol{w},\mathbf{L}) &:= -\big(\boldsymbol{w}, \boldsymbol{\nabla}\cdot(\mathbf{K}\mathbf{L}_i)\big)_{\Omega_i} + \big(\boldsymbol{w}, \mathbf{L}_i \boldsymbol{a}\big)_{\Omega_i} \\
a_{up}(\boldsymbol{w},p) &:= \big(\boldsymbol{w}, \boldsymbol{\nabla} p_i\big)_{\Omega_i} & a_{u\hat{u}}(\boldsymbol{w},\hat{\boldsymbol{u}}) &:= \langle \boldsymbol{w}, \tau_i \hat{\boldsymbol{u}} \rangle_{\partial\Omega_i}, \\
a_{LL}(\mathbf{G},\mathbf{L}) &:= \big(\mathbf{G}, \mathbf{L}_i\big)_{\Omega_i}, & a_{Lu}(\mathbf{G},\mathbf{u}) &:= \big(\boldsymbol{\nabla}\cdot \mathbf{G}, \boldsymbol{u}_i\big)_{\Omega_i}, \\
a_{L\hat{u}}(\mathbf{G},\hat{\boldsymbol{u}}) &:= \langle \mathbf{G}\boldsymbol{n}_i, \hat{\boldsymbol{u}} \rangle_{\partial\Omega_i}, & a_{pu}(q,\boldsymbol{u}) &:= \big(\boldsymbol{\nabla} q, \boldsymbol{u}_i\big)_{\Omega_i}, \\
a_{p\hat{u}}(q,\hat{\boldsymbol{u}}) &:= \langle q, \hat{\boldsymbol{u}}\cdot \boldsymbol{n}_i \rangle_{\partial\Omega_i}, & f_u(\boldsymbol{w}) &:= \big(\boldsymbol{w}, \boldsymbol{s}\big)_{\Omega_i}.
\end{aligned}
$$

References

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