MESHLESS METHODS IN DUAL ANALYSIS: THEORETICAL AND IMPLEMENTATION ISSUES

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Abstract. This paper presents a meshless implementation for the dual analysis of 2D linear elasticity problems. The derivation of the governing system of equations for the discretized equilibrated model is detailed. Crucial implementation issues of the equilibrated algorithm are described. It presents the inf-sup condition for particular cases of essential boundary conditions approximations when the space of Lagrange multipliers belongs to the space of domain nodal unknowns. Several examples with energy and stress verification are demonstrated.

1 INTRODUCTION

Meshless methods are well developed for modeling complex problems in solid mechanics. But most models are concentrated on usage of strain driven approach based on building the problem’s weak form either from the compatibility equation or from the potential energy [3, 12]. For the same problems, the alternative formulation which assumes that the equilibrium equations are satisfied and the compatibility equations may be violated can be used as well, but the use of the so-called stress driven approach using meshless approximations for the weak form discretization is restricted to very few papers, see [8].

The implementation of this approach is more demanding than the strain driven one due to the following facts: (i) higher order derivatives may appear in governing matrix
equations (if, for instance, stress functions are used), (ii) static boundary in most practical examples is larger than the one where kinematic boundary conditions are defined and (iii) some features of the problem can easily lead to numerical instability. This issues are not related with a specific approximation (FEM, Meshless, etc), but are common to all of them. In fact, it results in the relative unpopularity of this approach. When both models are used in the analysis of a particular problem, it is possible to perform the so-called dual analysis, introduced for the first time by Fraeij de Veubeke with respect to the conventional finite element method [6]. Regardless of the higher computational cost, the dual analysis can be efficiently used to obtain upper and lower bounds of some characteristic values of the system, typically, potential energy. Local quantities, like the stress and displacement fields, can also be verified [5].

The structure of this work is as follows. Section 2 presents the basic solid mechanics differential equations and two models required for dual analysis: compatible and equilibrated. The implementation issues, especially for the equilibrated model are discussed in details in Section 3. Two numerical examples are presented then in Section 4: the classical Timoshenko beam problem and the infinite plate with circular hole subjected to uniform tensile load. Finally, Section 5 collects our ideas and comments about theoretical and numerical results obtained during the research.

2 DUAL ANALYSIS

2.1 Linear elasticity equations

Consider a linear elastic plane domain, $\Omega$, with boundary, $\Gamma$, subjected to body forces, $\mathbf{B}$, in $\Omega$, distributed surface loads on the static boundary, $\Gamma_t$, and constrained with prescribed displacements on the kinematic boundary, $\Gamma_u$. As usual, we require that $\Gamma_u \cap \Gamma_t = \emptyset$ and $\Gamma_u \cup \Gamma_t = \Gamma$.

The relations governing the plane stress elasticity problem are outlined in the following. The compatibility equations in the domain are

$$\varepsilon = \nabla \mathbf{u} \text{ in } \Omega,$$

and on the kinematic boundary they are

$$\mathbf{u} = \mathbf{u} \text{ on } \Gamma_u,$$

The differential operator, which maps the displacements onto strains, is defined as

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$$
The equilibrium equations in the domain are
\[ \nabla^T \sigma + \bar{b} = 0 \text{ in } \Omega, \] (4)
and on the static boundary they are
\[ N \sigma = \bar{t} \text{ on } \Gamma_t, \] (5)
Here \( N \) is the projection of \( \nabla \) in the boundary, gathering the outward normal components, \textit{i.e.},
\[ N = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \] (6)
The constitutive relation is
\[ \sigma = D \varepsilon, \] (7)
where \( D \) is a constitutive tensor, which for two dimensional plane stress case is
\[ D = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \] (8)
Let us also introduce the potential energy \( \Pi = \Pi(u) \) and the complementary potential energy \( \Pi^* = \Pi^*(\sigma) \), both will be outlined below. It can be shown [7] that, under appropriate conditions, the exact value of the potential energy of the system is bounded by these two values:
\[ \Pi < \Pi_{\text{exact}} < \Pi^* \] (9)

2.2 Equilibrated model
We start from the equilibrated model, making a short review of the compatible one afterwards. To derive the required weak form let us define a complementary potential energy [7]:
\[ \Pi^*(\sigma) = \int_{\Omega} U^*_0(\sigma) \, d\Omega - \int_{\Gamma_u} t^T \bar{u} \, d\Gamma_u, \quad \text{where } U^*_0(\sigma) = \frac{1}{2} \sigma^T D^{-1} \sigma \] (10)
The static and kinematic boundary conditions are the essential and natural ones, respectively, in this case. The natural ones are explicitly added as a second term in (10) while the essential boundary conditions can be imposed in various ways [9]. In the present work Lagrange multipliers are added into the complementary potential energy functional (10), resulting in the following augmented complementary potential energy functional:
\[ \Pi^*(\sigma, u) = \Pi^*(\sigma) - \int_{\Gamma_t} u^T(t - \bar{t}) \, d\Gamma_t \] (11)
From the physics of the problem it follows that Lagrange multipliers represent displace-
ments on the static boundary. The notation in (11) was chosen to reflect this fact.

The augmented weak form of the problem can be recovered by taking variation of (11)
and equating it to zero:

$$
\int_{\Omega} \delta \sigma^T D^{-1} \sigma \, d\Omega - \int_{\Gamma_u} (N \delta \sigma)^T \bar{u} \, d\Gamma_u - \int_{\Gamma_t} \delta \sigma^T \bar{N} \, d\Gamma_t = 0
$$

(12)

To derive (12) we took into account that $\delta \bar{u} = 0$ and used (6) as well. Notice that the (12)
doesn’t imply any special requirements for the stresses, besides the ones arise from (4).

To make this equilibrium condition satisfied all over the domain we consider stresses to
be expressed by means of the Airy stress function [13]:

$$
\sigma = (\Delta \Psi + \bar{V}) I - \nabla \nabla^T \Psi,
$$

(13)

where $\nabla$ is the gradient, $\Delta$ is the Laplacian $\nabla^T \cdot \nabla$, $I$ is the identity matrix and $\bar{V}$ defines
the potential of the domain body forces as follows:

$$
\bar{b} = -\nabla \bar{V}
$$

(14)

For two dimensional case (13) can be written in a simpler manner:

$$
\sigma = \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{pmatrix}
= \begin{pmatrix}
\Psi_{,yy} \\
\Psi_{,xx} \\
-\Psi_{,xy}
\end{pmatrix} + \gamma \bar{V}, \text{ where } \gamma = \begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}
$$

(15)

Let us introduce a discretization for the Airy stress function and its variation in the
form:

$$
\Psi^h = \sum_{i=1}^{n} \phi_i(x) \Psi_i
$$

$$
\delta \Psi^h = \sum_{i=1}^{n} \phi_i(x) \delta \Psi_i
$$

(16)

or using matrix notation:

$$
\Psi^h = \Phi \Psi
$$

$$
\delta \Psi^h = \Phi \delta \Psi
$$

(17)

where $\phi_i$ is an arbitrary domain approximation function. With respect to the latter
equation and also implying that $\delta \bar{V} = 0$ we obtain the discretized forms of the domain
stresses and theirs variations for two dimensional case:

$$
\sigma^h = C \Psi + \gamma \bar{V}, \text{ where } C = \begin{pmatrix}
\Phi_{,yy} \\
\Phi_{,xx} \\
-\Phi_{,xy}
\end{pmatrix}
$$

(18)
Also let us introduce an approximation for the Lagrange multipliers and theirs variations:

\[ u^h = Lu \]
\[ \delta u^h = L\delta u \]  

(19)

Until now we have not told anything about functions used for both approximations (17) and (19), but in fact there are some limitations exist on their choice. We will highlight this question later (see Section 3.3).

Let substitute now (18) and (19) into (12). After some simplifications it is possible to obtain

\[
\delta \Psi^T \int_{\Omega} \Phi^T D^{-1} \Phi d\Omega \Psi + \delta \Psi^T \int_{\Omega} \Phi^T D^{-1} \gamma \nabla d\Omega - \\
\delta \Psi^T \int_{\Gamma_u} \Phi^T N^T \bar{u} d\Gamma_u - \delta u^T \int_{\Gamma_t} L^T N \Phi d\Gamma_t \Psi + \\
\delta u^T \int_{\Gamma_t} L^T \tilde{t} d\Gamma_t - \delta \Psi^T \int_{\Gamma_t} \Phi^T N^T L d\Gamma_t u - \delta u^T \int_{\Gamma_t} L^T N \gamma \nabla d\Gamma_t = 0
\]

(20)

Since the weak form (20) must hold for \( \forall \delta \Psi \in \mathbb{U} \) and \( \forall \delta u \in \mathbb{V} \) we finally can write the governing system of equations:

\[
\begin{bmatrix}
F & G \\
G^T & 0
\end{bmatrix}
\begin{bmatrix}
\Psi \\
u
\end{bmatrix} =
\begin{bmatrix}
d \\
f
\end{bmatrix}
\]

(21)

where

\[
F_{IJ} = \int_{\Omega} \Phi^T_I D^{-1} \Phi_J d\Omega
\]

(22a)

\[
G_{IK} = -\int_{\Gamma_t} \Phi^T_I N^T L_K d\Gamma_t
\]

(22b)

\[
d_I = \int_{\Gamma_u} \Phi^T_I N^T \bar{u} d\Gamma_u - \int_{\Omega} \Phi^T_I D^{-1} \gamma \nabla d\Omega
\]

(22c)

\[
f_K = \int_{\Gamma_t} L^T_K N \gamma \nabla d\Gamma_t - \int_{\Gamma_t} L^T_K \tilde{t} d\Gamma_t
\]

(22d)

and

\[
L_K = \begin{bmatrix} L_K & 0 \\ 0 & L_K \end{bmatrix}
\]

(23)

2.3 Compatible model

Comparing to the model presented above this one must satisfy the compatibility relations, instead of the equilibrium ones. To derive the required weak form let us define a
potential energy:

$$\Pi(u) = \int_{\Omega} U_0(u) \, d\Omega - \int_{\Gamma} u^T \mathbf{t} \, d\Gamma, \quad \text{where } U_0(u) = \frac{1}{2} u^T \mathbf{D} u \quad (24)$$

The kinematic and static boundary conditions are the essential and natural ones, respectively, in this case. The natural ones are explicitly added as a second term in (24) and the essential boundary conditions can be imposed in various ways [9]. In the present work Lagrange multipliers are added into the potential energy (24), resulting the following augmented potential energy functional:

$$\Pi(u, t) = \Pi(\sigma) - \int_{\Gamma_u} t^T (u - \bar{u}) \, d\Gamma_u \quad (25)$$

From the physics of the problem it follows that Lagrange multipliers represent tractions on the kinematic boundary. The notation in (25) was chosen to reflect this fact.

The augmented weak form of the problem can be recovered by taking variation of (25) and equating it to zero [12]:

$$\int_{\Omega} \delta \varepsilon^T \mathbf{D} \varepsilon \, d\Omega - \int_{\Omega} \mathbf{b}^T \delta u \, d\Omega - \int_{\Gamma} \mathbf{t}^T \delta u \, d\Gamma - \int_{\Gamma_u} \delta t^T (u - \bar{u}) \, d\Gamma_u - \int_{\Gamma_u} t^T \delta u \, d\Gamma_u = 0 \quad (26)$$

The approximants for the domain displacements and unknown boundary tractions are defined in the similar manner as it was done above for the equilibrated model, being the only difference now the fact that the domain approximation (for the displacements) has two degrees of freedom instead of one, therefore the $$\Phi_K$$ takes form:

$$\Phi_K = \begin{bmatrix} \Phi_K & 0 \\ 0 & \Phi_K \end{bmatrix} \quad (27)$$

The displacements $$u^h$$ and their variations $$\delta u^h$$ are approximated with certain interpolants:

$$u^h = \Phi u \quad (28)$$

The essential boundary tractions $$\mathbf{t}^h$$ have the approximation defined by matrix $$\mathbf{L}$$, which will be detailed below

$$\mathbf{t}^h = \mathbf{L} \mathbf{t} \quad (29)$$

The equation (26) leads to the following system:

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix} \quad (30)$$
where

\[ B = \nabla \Phi_I \] (32)

3 IMPLEMENTATION ISSUES

3.1 Domain approximation functions

Let \( \mathbf{x}_i \) be the coordinates of a particle \( i \) in a domain \( \Omega \) containing \( n \) nodes. Then the multiple fixed least-squares\(^1\) approximation \( f^h(\mathbf{x}) \) for an arbitrary function \( \mathbf{f}(\mathbf{x}) \) is [3, 12]:

\[ f^h(\mathbf{x}) = \Phi(\mathbf{x})\mathbf{f}, \text{ where } \Phi(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x}) \] (33)

Here we denoted

\[ \mathbf{A}(\mathbf{x}) = \mathbf{P}^T \mathbf{V}(\mathbf{x}) \mathbf{P} \text{ and } \mathbf{B}(\mathbf{x}) = \mathbf{P}^T \mathbf{V}(\mathbf{x}), \] (34)

where matrix \( \mathbf{P} \) contains values of basis functions \( \mathbf{p}(\mathbf{x}) \)\(^2\) for each particle of a domain:

\[ \mathbf{P} = [\mathbf{p}(\mathbf{x}_1) \, \mathbf{p}(\mathbf{x}_2) \, \ldots \, \mathbf{p}(\mathbf{x}_n)]^T \] (35)

and

\[ \mathbf{V} = \text{diag}[w(\mathbf{x}_1 - \mathbf{x}), w(\mathbf{x}_2 - \mathbf{x}), \ldots, w(\mathbf{x}_n - \mathbf{x})] \] (36)

To specify \( \mathbf{V} \) we need to define a weight function. For this particular case we chose quintic spline function with circular support:

\[ w = \begin{cases} 1 - 10r^2 + 20r^3 - 15r^4 + 4r^5 & \text{if } r \leq 1 \\ 0 & \text{if } r > 1 \end{cases} \] (37)

where \( r_i \) is the support radius for the particle \( i \).

\(^1\)Here we use the Multiple Fixed Least-Squares approximation [11] and distinguish it from the Moving Least-Squares approximants, but it can be shown that all the conclusions we will obtain below holds for the MLS case as well.

\(^2\)For our case we define them as polynomials \( x^k \) of order \( k \), but the basis can be formed of any linearly independent functions.
3.2 Computation of the support radius

According to the properties of approximation functions (33), the support size influences significantly the sparsity of stiffness or flexibility matrix and, consequently, the computational cost. Hence, from the practical point of view, it’s desirable to choose smaller support but its minimal size is bounded by the requirement of generating a non-singular moment matrix $A$ in (33). Consider the usage of polynomial basis of order $k > 0$. To avoid matrix ill-conditioning in this case supports must satisfy the following conditions [12]:

- each point of interest must be covered at least by $\frac{1}{2}(k+1)(k+2)$ supports from its neighbourhood,
- for 2D case a certain point of interest must be covered in each of at least two non-collinear directions by more than $m$ supports.

Let us choose now the order of basis to be used for both models. For the compatible model to reproduce constant stress field we have to use linear basis since stresses are first derivatives of displacements. But to reproduce it with the equilibrated solution we must use at least quadratic basis, which results in the loss of sparsity. For instance, to obtain the exact solution of Timoshenko beam example (see Section 4.1) we have to use 3rd order basis for the compatible model and 4th order for the equilibrated one.

3.3 Lagrange multipliers approximation functions

To approximate Lagrange multipliers we use two types of approximants:

- Lagrange polynomials of a prescribed order:

  $$L(s) = \sum_{j=0}^{k} L_j(s), \text{ where } L_j(s) = y_j \prod_{0 \leq p \leq k \atop p \neq j} \frac{s - s_p}{s_j - s_p}$$  

- the trace on the boundary of the domain approximation functions, i. e.

  $$L = \Phi$$

These two approaches have completely different fundamental behaviors. The first one implies that we use different spaces to approximate values within the domain and on the essential boundary. As it will be exposed, this results in numerical stability thanks to the LadyzhenskayaBabuškaBrezzi (LBB)-condition fulfilled in this case, but the drawback of this approach is that we can’t impose boundary conditions exactly.

On the other hand it can be shown [12] that the exact imposition of the essential boundary conditions requires that both spaces for the approximation function inside a
domain and on a boundary must coincide. But in practice MFLS approximants can’t be used explicitly in this manner. For instance, consider the domain on figure 1 clamped on the left side and subjected to a unit tensile load uniformly distributed on the right side. Material parameters are Young’s modulus $E = 1 \cdot 10^3$ and Poisson’s ratio $\nu = 0.3$.

![Figure 1: Tension test. Problem statement.](image1)

![Figure 2: Tension test. Particles distribution.](image2)

This problem was analysed with the two possible approximants for the essential boundary and with the compatible model. On the second solution of the figure 3 it can be seen, that zero value displacements propagate inside a domain, clamping in the $y$-direction not only the boundary but also the area where the displacements are approximated by the particles which supports intersect the boundary. It is visible that exactly two lines of particles intersect the boundary and the area bounded by the second line is completely locked (figure 3).

![Figure 3: Comparison of Lagrange polynomials and MFLS approximation functions for tractions approximation.](image3)

To explain this result let us write the discrete bilinear form of (30):

$$
\begin{align*}
\{a(u_h, v_h) + b(v_h, \phi_h) = f(v_h) \quad \forall v_h \in V_h \\
b(u_h, \psi_h) = g(\psi_h) \quad \forall \psi_h \in W_h
\}
\end{align*}
$$

(40)

where $u_h$ and $\phi_h$ forms arbitrary solution of (40). For this case, the LBB condition reads [1, 2]:

$$
\inf_{\psi_h \in W_h} \sup_{v_h \in V_h} \frac{b(v_h, \psi_h)}{\|v_h\|_1 \|\psi_h\|_0} \geq \beta > 0
$$

(41)
According to [1, 4] the inf-sup condition can be rewritten with matrix notation:

\[
\inf_{\psi_h} \sup_{v_h} \frac{-V_h^T G \Psi_h}{\|v_h\|_1 \|\psi_h\|_0} \geq \beta > 0 \tag{42}
\]

where vectors \( V_h \) and \( \Psi_h \) must satisfy the solvability conditions for (30) [1, p. 313].

Here we also implied that the matrix \( G \) is related to the bilinear form \( b(v_h, \psi_h) \) as follows [1ref]:

\[
b(v_h, \psi_h) = -V_h^T G \Psi_h \tag{43}
\]

Using the same approximation functions for both kind of variables we rewrite (31b):

\[
G = -\int_\Gamma (\Phi_m^T \Phi_m) \, d\Gamma \tag{44}
\]

Here \( m \) subscript refers to domain approximation functions which contribute to the boundary, i.e. \( m \) is a number of domain particles whose supports intersect the boundary. Therefore the approximation space \( W_h \) of the Lagrange multipliers containing traces of the domain approximation functions is a subspace of \( V_h \). Due to this fact, the size of matrix \( G \) is \( n \times m \) (where \( n \) is a total number of particles in the domain, but only the \( m \times m \) part is different from zero. Therefore we can write \( G \) as follows:

\[
G = \begin{bmatrix} -T \\ 0 \end{bmatrix}, \text{ where } T = \int_\Gamma (\Phi_m^T \Phi_m) \, d\Gamma \tag{45}
\]

To keep the multiplication \( V_h^T G \) correct we can similary subdivide the vector of nodal values \( V_h \) into 2 parts:

\[
V_h = [V_1 \ V_2] = [\Psi_h \ V_2] \tag{46}
\]

Here we also implied that space \( W_h \in V_h \) according to previous assumption to substitute \( V_1 \).

Now using (45) and (46) and since \( V_1 \) refers to nodal values of particles which intersect the boundary, we can simplify the numerator of (42):

\[
V_h^T G \Psi_h = [V_1^T \ V_2^T] \begin{bmatrix} -T \\ 0 \end{bmatrix} \Psi_h = -\Psi_h^T T \Psi_h \tag{47}
\]

The denominator of (42) contains norms which can be introduced in a different manner. We follow the idea proposed in [1, p. 301] to define them for the vector values as an equivalent of the Sobolev \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \) norms respectively:

\[
\|v\|_1 = \sum_{i,j} \left| \frac{\partial v_i}{\partial x_j} \right|_{L^2(\Omega)} \tag{48}
\]

Notice the difference between terms \( v_h \) and \( V_h \): the former is a test function from the \( V_h \) space and the later forms the space of vectors that satisfy \( G^T V_h = 0 \), i.e. represents the kernel of \( G^T \). The similar remark should be noted for \( \psi_h \) and \( \Psi_h \).
and

$$\| \psi \|_0 = \sum_i |\psi_i|_{L^2(\Omega)}$$  (49)

The later equation can be rewritten as:

$$\left( |\psi_h|_0 \right)^2 = \int_\Omega \left( \sum_{i=1}^3 (\psi_i)^2 \right) d\Omega$$  (50)

The discrete solution space $\psi_h$ is approximated using the vector of nodal values $\Psi_h$:

$$\psi_h = \Phi_m \Psi_h$$  (51)

Substituting the latter expression into (50) we obtain:

$$\left( |\psi_h|_0 \right)^2 = \int_\Gamma \left( (\Phi_m \Psi_h)^T (\Phi_m \Psi_h) \right) d\Gamma = \Psi_h^T \int_\Gamma (\Phi_m^T \Phi_m) d\Gamma \Psi_h = \Psi_h^T T \Psi_h$$  (52)

Let us notice, that since Lagrange multipliers are defined along the boundary, we consider the integration domain in (52) to be $\Gamma$.

After all deductions we can rewrite the inf-sup condition for this particular case:

$$\inf_{\Psi_h} \sup_{\nu_h} \frac{\Psi_h^T T \Psi_h}{(\Psi_h^T T \Psi_h)^{\frac{1}{2}} \|\nu_h\|} \geq \beta > 0$$  (53)

what results in:

$$\inf_{\psi_h \in \mathcal{W}_h} \sup_{\nu_h \in \mathcal{V}_h} \frac{\|\psi_h\|}{\|\nu_h\|} \geq \beta > 0$$  (54)

This expression means that if we use traces of the internal variables approximation functions for Lagrange multipliers we must make an interacting approximation function to bring the maximum value on a boundary, i.e., propagating inside a domain the approximation function should decrease. This property is perfectly fulfilled by the particles which are on the boundary (or very close to). Moreover it is now clear why Finite Elements do not meet this kind of problems: since the FEM shape functions fulfil the Kroneker $\delta_{ij}$ property, the LBB condition (54) is automatically satisfied (figure 4).

To overcome the described difficulty some constraints on the boundary should be relaxed. It can be done in a standard manner by means of the Moore-Penrouse pseudo-inverse to solve the governing system of equations. In fact, this procedure introduces some error in essential boundary conditions incorporation, but eliminates matrix ill-conditioning.
Figure 4: Inf-sup condition for the FEM and MFLS approximation functions for the boundary approximation.

3.4 Flexibility matrix natural ill-conditioning

Let us notice that both the stiffness matrix in (30) and the flexibility matrix in (21) are rank deficient since essential boundary conditions are only imposed afterwards. In the compatible model they are imposed directly on the displacements which are domain unknowns in this case, but for the equilibrated model they are imposed on the stresses, which don’t coincide with domain unknowns (the weights of the Airy stress function approximation). That means that for the stress function three rigid body modes are not restrained even once the boundary conditions are imposed. To eliminate this ill-conditioning we have to impose three extra conditions directly on the values of the Airy function. The ones can be set to any arbitrary values in any three noncolinear particles, since if \( \Psi(x, y) \) is a solution of (21) then \( \Psi(x, y) = \Psi(x, y) + ax + by + c \) is also a solution of a system, where \( a, b \) and \( c \) are arbitrary constants, since

\[
\Psi(x, y)_{xx} = \Psi(x, y)_{xx}, \quad \Psi(x, y)_{yy} = \Psi(x, y)_{yy}, \quad \Psi(x, y)_{xy} = \Psi(x, y)_{xy}
\]  

These restrictions are also imposed by means of Lagrange multipliers:

\[
\lambda_i^\Psi (\Psi_i - \overline{\Psi_i}) = 0, \quad \text{where} \quad i = \{k, l, m\}, \quad 0 < i \leq N,
\]  

where \( N \) is a total number of particles in a domain.

3.5 Linear dependencies on the essential boundary

When imposing essential boundary conditions, one should take care of possible linear dependencies which arise in the connection point of two adjacent boundaries. This problem appears in both models. We present it for compatible one only.

On figure 5 two boundaries with prescribed boundary conditions are connected at point \( A \). Since this point belongs to discretization of both boundaries its displacements are constrained twice and Lagrange multipliers vectors \( \lambda_A^{\Gamma_1} \) and \( \lambda_A^{\Gamma_2} \) turn out to be linearly
depended. To release this constraint we add one more Lagrange multiplier in a following manner:

\[ \alpha \left( \lambda_{A_{\Gamma_1}} - \lambda_{A_{\Gamma_2}} \right) = 0 \]  

(57)

We also note that it is necessary to ensure the admissibility of nonhomogeneous boundary conditions. For example, at point \( A \), the horizontal displacement in \( \Gamma_{t_1} \) must coincide with the one in \( \Gamma_{t_2} \). The complementary conditions also apply to the equilibrated model.

4 Numerical examples

4.1 Timoshenko beam problem

Consider a beam subjected to a parabolic tip load as represented in figure 6. The dimensions of the beam are \( L = 48, h = 12 \). The cross section of the beam is considered to have unit width. Material parameters are Young’s modulus \( E = 3 \cdot 10^7 \) and Poisson’s ratio \( \nu = 0.3 \).

The imposed tangential stress on the right edge is

\[ \tau_{xy} = -\frac{P}{2I} \left( y^2 - \frac{h^2}{4} \right), \text{ where } I = \frac{h^3}{12}. \]  

(58)
Hence, the total distributed load acting along the tip boundary renders:

\[ P = \int_{-h/2}^{h/2} \tau_{xy} dy = 1000 \]  

(59)

The left edge of the beam is subjected to the kinematic boundary conditions defined by the exact solution for displacements:

\[ u_x = -\frac{Py}{6EI} \left\{ (6L - 3x)x + (2 + \nu) \left( y^2 - \frac{h^2}{4} \right) \right\} \]

\[ u_y = \frac{P}{6EI} \left\{ 3\nu y^2(L - x) + (4 + 5\nu)\frac{h^2x}{4} + (3L - x)x^2 \right\} \]  

(60)

The computation was performed with different approximations (see figure 7, for instance) and different orders of basis \( k \). The exact solution of this problem gives the potential energy\(^4\)

\[ \Pi_{\text{exact}} = \frac{2LP^2(3h^2 + 5L^2 + 3h^2\nu)}{5h^3E} = 4.474667 \]  

(61)

\(^4\)In fact for this problem the exact solution can be obtained by means of both models if we use 3rd and 4th order polynomials respectively.
and according to the results on figure 8 both model perfectly bounds this value.

The energy convergence slopes (see figure 8) for the compatible solution can be estimated and numerical solutions give the predicted values, but for the equilibrated model the authors are not aware of such assessments. The results presented indicate the complexity of the model.

4.2 Hole in an infinite plate

Consider the problem of the infinite plate [10] with a central circular hole subjected to an unidirectional tensile load in $x$ direction. Plane stress conditions are assumed and Young’s modulus is $E = 1 \cdot 10^3$ and Poisson’s ratio is $\nu = 0.3$. Only the upper right quadrant of the plate is modeled due to symmetry (figure 9). To this end we apply on the right and top sides of the plate the stresses which correspond to exact solution of the problem:

\[
\begin{align*}
\sigma_{xx}(r, \theta) &= 1 - \frac{a^2}{r^2} \left\{ \frac{3}{2} \cos 2\theta + \cos 4\theta \right\} + \frac{3a^4}{2r^4} \cos 4\theta \\
\sigma_{yy}(r, \theta) &= -\frac{a^2}{r^2} \left\{ \frac{1}{2} \cos 2\theta - \cos 4\theta \right\} - \frac{3a^4}{2r^4} \cos 4\theta \\
\sigma_{xy}(r, \theta) &= -\frac{a^2}{r^2} \left\{ \frac{1}{2} \sin 2\theta + \sin 4\theta \right\} + \frac{3a^4}{2r^4} \sin 4\theta
\end{align*}
\]

where $r, \theta$ are polar coordinates and $\theta$ is measured from the positive $x$ axis counterclockwise.

The exact value of potential energy is $\Pi_{exact} = 0.0129754$. Both models demonstrate similar convergence rates (figure 11) comparing to the ones given by the previous example.
To show that the stresses in both cases coincide as well we provide figure 12, where the stresses $\sigma_{xx}$ in the left edge of the plate are shown. For this case the discretization from figure 13 was considered and 2nd and 4th order bases were used to build compatible and equilibrated solutions respectively. Since the displacements in $x$ direction are prevented on the left edge due to symmetry conditions we also can obtain the values of Lagrange multipliers which correspond to tractions $t_x$ in compatible model. This graph perfectly coinciding with the values of stresses computed by both models is presented on figure 12.

5 CONCLUSIONS

We demonstrated successfully the possibility to use dual analysis approach by means of meshless methods with multiple fixed least-squares approximations used as a core for the discretization process. Of course, the equilibrated model meets a lot of implementation issues which can be easily avoided in the compatible solution. But we can outline the key requirements for its robust implementation:

- The domain requires a good particles discretization even for simple cases in order to avoid ill-conditioning if the $A$ matrix used to build MFLS approximation functions. This problem arise due to high order of the basis we use to build the equilibrated solution;
- The natural ill-conditioning of the flexibility matrix should be explicitly considered;

- Some dependencies may arise on the essential boundary. They should be eliminated, for instance, by means of the introduction of extra Lagrange multipliers. In fact, this problem also appears in the compatible solution, but since the kinematic boundary is usually shorter than the static one, the equilibrated model is much more prone to this kind of problems. Care should also be taken to ensure the admissibility of the applied loads.

Also we presented an explanation of why the domain MFLS functions should’t be used for approximants of the Lagrange multipliers imposed on essential boundaries. It is based on the inf-sup condition for mixed models. Conventional finite elements are insensitive to this limitation due to Kronecker-Delta property relevant for their shape functions. It induces us to use totally different functions space for the boundary approximation.

Once all the limitations are known the dual analysis can be performed efficiently. Of course the stress model can not replace the displacement one due to the reasons mentioned above but it can serve as a natural verification method for a deformation energy and stress values bringing more confidence in the obtained solution.

REFERENCES


