A new equilibrated residual method: improving accuracy and efficiency of flux-free error estimates in two and three dimensions

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Guaranteed accurate and efficient bounds

The finite element method is a basic tool in engineering design and is crucial to certify the quality of the results.

A lot of work has been done to provide certificates of the approximate solution, i.e. obtain guaranteed bounds in which the exact solution lies (either in energy norm or in QoI).

GOAL: $|||e||| \leq \eta$ or $s^- \leq \ell^{\mathcal{O}}(e) \leq s^+$

The desired qualities of a posteriori estimators are:

- CERTIFICATION: they should provide guaranteed/strict bounds
- ACCURACY: they should be accurate (good effectivities)
- COST: they should be cheap (involve small local problems)

Guaranteed accurate and efficient bounds

CERTIFICATION		
complementary energy dual formulation for the error	+	implicit error estimators involving only local problems

Guaranteed accurate and efficient bounds

CERTIFICATION

complementary energy + implicit error estimators

dual formulation for the error — involving only local problems

Hybrid-flux estimators
equilibrated
→ **CHEAPER**

Flux-free estimators
stars/subdomain [PDH2006]
→ **MORE ACCURATE**

Guaranteed accurate and efficient bounds

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MORE ACCURATE

CHEAP + ACCURATE

EXPLICIT Flux-free estimator

Model problem

Reaction-diffusion equation:
$$\begin{aligned} -\Delta u + \kappa^2 u &= f && \text{in } \Omega, \\ u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}, \\ \nabla u \cdot \boldsymbol{n} &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

Weak form: find $u \in \mathcal{U}$ such that
$$\underbrace{\int_\Omega \left(\nabla u \cdot \nabla v + \kappa^2 uv \right) \, d\Omega}_{a(u,v)} = \underbrace{\int_\Omega fv \, d\Omega + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} v \, d\Gamma}_{\ell(v)} \qquad \forall v \in \mathcal{V}.$$

Finite element approximation: find $u_h \in \mathcal{U}^h$ such that
$$a(u_h, v) = \ell(v) \qquad \text{for all } v \in \mathcal{V}^h.$$

triangular mesh + linear elements

Error equations: find $e = u - u_h \in \mathcal{V}$ such that
$$a(e, v) = \ell(v) - a(u_h, v) = R(v) \qquad \text{for all } v \in \mathcal{V}.$$

Guaranteed error bounds

The complementary energy approach allows to overestimate $|||e|||$
approach introduced by Fraeijs de Veubeke in 1964

$$a(e,v) = \int_\Omega \left(\boldsymbol{\nabla} e \cdot \boldsymbol{\nabla} v + \kappa^2 e v\right) d\Omega = R(v) \qquad \text{for all } v \in \mathcal{V}$$

$$\int_\Omega \left(\boldsymbol{q} \cdot \boldsymbol{\nabla} v + \kappa^2 r v\right) d\Omega = R(v) \qquad \text{for all } v \in \mathcal{V}$$

new error unknowns ($\boldsymbol{q}$, r)

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new error unknowns

Dual formulation:

Any pair of dual estimates $(\boldsymbol{q}, r)$ such that

$$\int_\Omega \left(\boldsymbol{q} \cdot \nabla v + \kappa^2 r v\right) d\Omega = R(v) \qquad \text{for all } v \in \mathcal{V}$$

provide an upper bound for the energy norm of the error

$$|||e|||^2 = \int_\Omega \left(\nabla e \cdot \nabla e + \kappa^2 e^2\right) d\Omega \leq \int_\Omega \left(\boldsymbol{q} \cdot \boldsymbol{q} + \kappa^2 r^2\right) d\Omega$$

complementary energy

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Guaranteed error bounds

Optimal choice: $(\boldsymbol{q}, r) = (\boldsymbol{\nabla} e, e)$

$$|||e|||^2 = \int_\Omega \left(\boldsymbol{q}\cdot\boldsymbol{q} + \kappa^2 r^2\right) d\Omega$$

Very accurate but expensive:

compute piecewise polynomial $(\boldsymbol{q}, r)$ solving a GLOBAL problem

Accurate but cheaper:

compute piecewise polynomial $(\boldsymbol{q}, r)$ solving LOCAL problems

Global problem $\Longrightarrow$ domain decomposition!

$$\int_\Omega \left(\boldsymbol{q}\cdot\boldsymbol{\nabla} v + \kappa^2 r v\right) d\Omega = R(v)$$

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Hybrid-flux / Flux-free

Guaranteed error bounds

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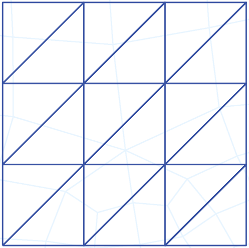

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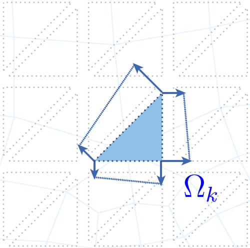

Global problem

$$\int_{\Omega} \left(\boldsymbol{q} \cdot \boldsymbol{\nabla} v + \kappa^2 r v \right) d\Omega = R(v)$$

Hybrid-flux

$$\int_{\Omega_k} \left(\boldsymbol{q}_k \cdot \boldsymbol{\nabla} v + \kappa^2 r_k v \right) d\Omega = R_k(v) + \int_{\Omega_k} g_k v \, d\Gamma$$

$$\boldsymbol{q}|_{\Omega_k} = \boldsymbol{q}_k \quad , \quad r|_{\Omega_k} = r_k$$

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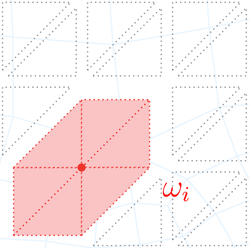

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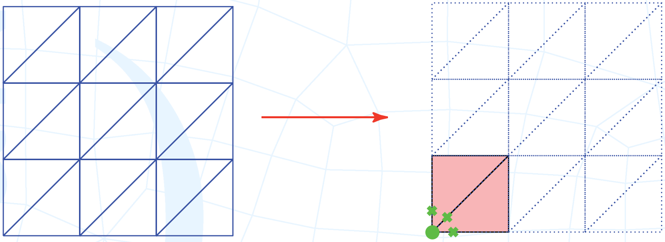
Flux-free

$$\int_{\omega_i} \left(\boldsymbol{q}^i\cdot\boldsymbol{\nabla} v + \kappa^2 r^i v\right) d\Omega = R(\phi_i v)$$

$$\boldsymbol{q} = \sum_{i=1}^{n_{\mathrm{np}}} \boldsymbol{q}^i \quad , \quad r = \sum_{i=1}^{n_{\mathrm{np}}} r^i$$

Hybrid-flux / equilibrated error estimates

STEP 1: loop in nodes to compute the equilibrated tractions g_k



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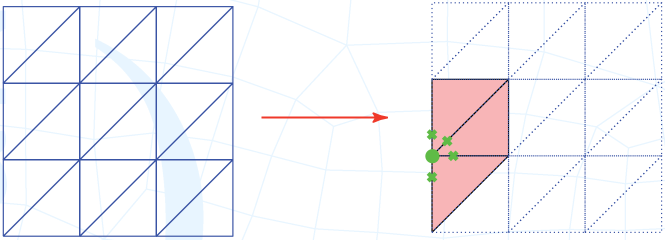

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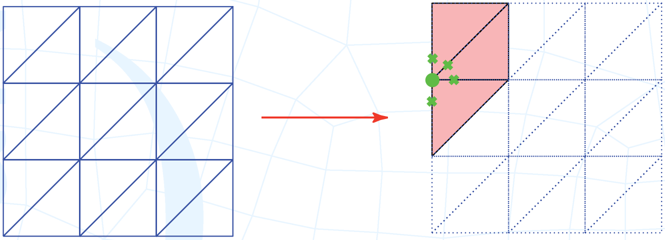

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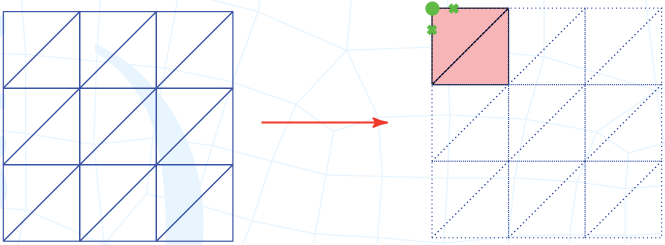

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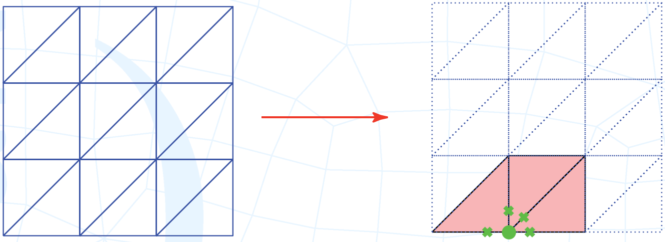

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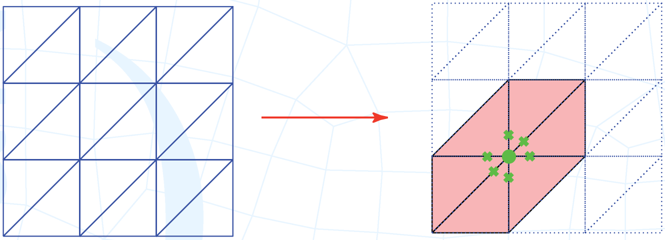

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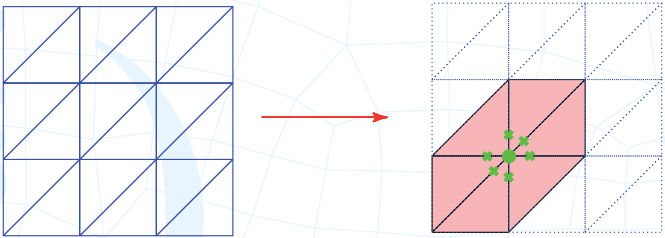
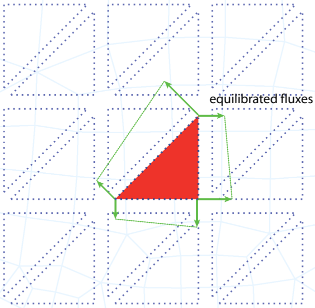


STEP 2: loop in elements to compute the dual fluxes

$$(\boldsymbol{q}_k, r_k)$$

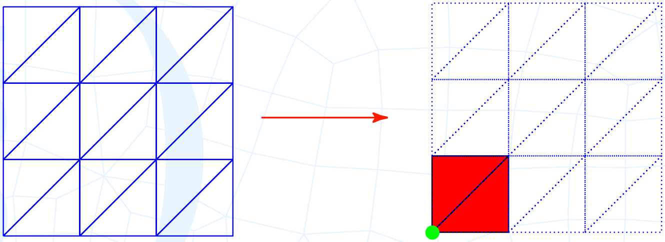
at each element Ω_k independently



Flux-free error estimates

STEP 1: loop in nodes to compute the dual fluxes in the stars

$$(\boldsymbol{q}^i, r^i) \quad \text{in} \quad \omega_i \quad \text{(patch of elements)}$$



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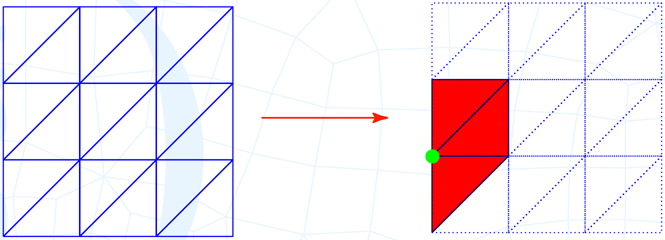

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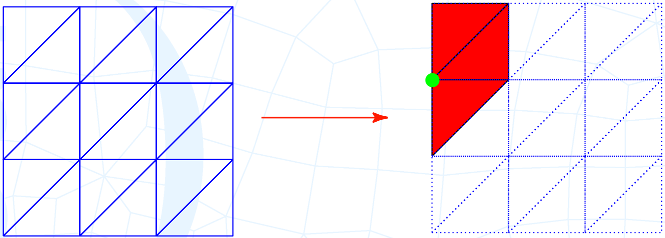

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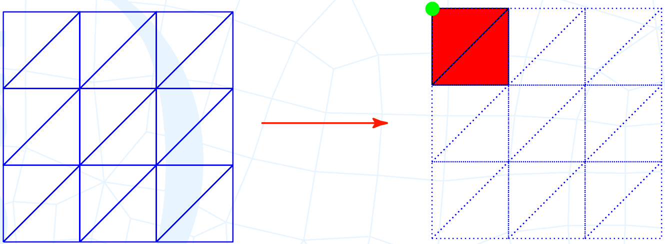

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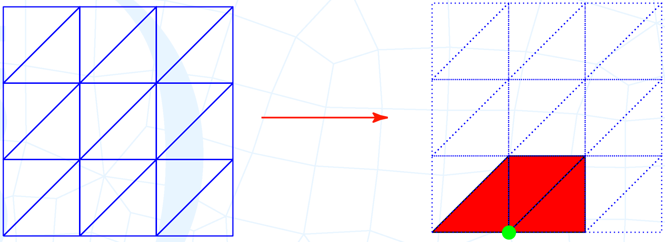

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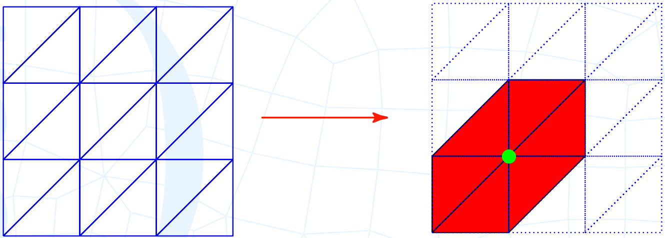

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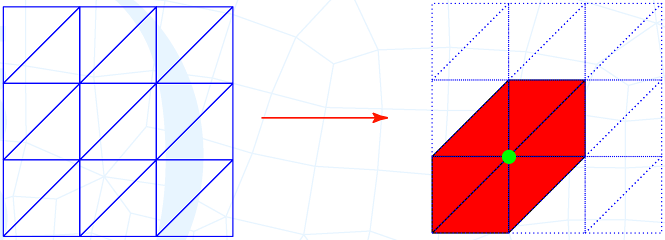
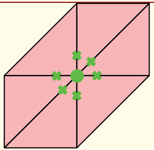
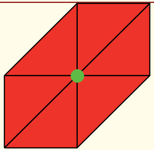
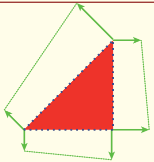


STEP 2: add all the local contributions and compute the norm

$$\boldsymbol{q} = \sum_{i=1}^{n_{\text{np}}} \boldsymbol{q}^i \quad , \quad r = \sum_{i=1}^{n_{\text{np}}} r^i$$

Computational cost overview

	Equilibrated	Flux-free
Loop on nodes	 DOF: one per edge of ω_i	 DOF: dof of $(\boldsymbol{q}_k^i, r_k^i)$ $\times$ elements of ω_i
Loop on elements	 DOF: dof of $(\boldsymbol{q}_k, r_k)$	

Computational cost overview

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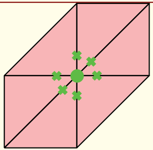
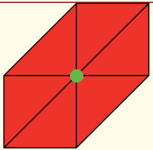
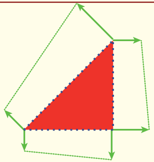
Loop on nodes	 DOF: one per edge of ω_i	 DOF: dof of $(\boldsymbol{q}_k^i, r_k^i)$ $\times$ elements of ω_i
Loop on elements	 DOF: dof of $(\boldsymbol{q}_k, r_k)$	

2D , q=2 → 18
2D , q=3 → 30

Computational cost overview

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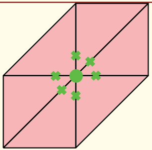
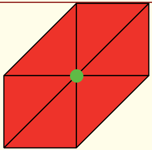
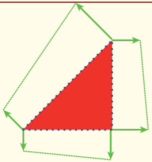
Loop on nodes	 DOF: one per edge of ω_i	 DOF: dof of $(\boldsymbol{q}_k^i, r_k^i)$ $\times$ elements of ω_i
Loop on elements	 DOF: dof of $(\boldsymbol{q}_k, r_k)$	☹ higher cost ☺ better accuracy

2D , q=2 $\rightarrow$ 18
2D , q=3 $\rightarrow$ 30

New guaranteed, accurate and cheap error estimate (EE)

Goal: decompose the global problem into stars ω_i

$$\int_\Omega \left(\boldsymbol{q} \cdot \boldsymbol{\nabla} v + \kappa^2 r v\right) d\Omega = R(v) \quad \forall v \in \mathcal{V}$$

minimizing the global complementary energy

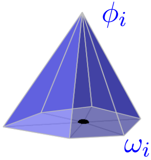
$$\int_\Omega \left(\boldsymbol{q} \cdot \boldsymbol{q} + \kappa^2 r^2\right) d\Omega$$

Local problems: $\boldsymbol{q} = \sum_{i=1}^{n_{\mathrm{np}}} \boldsymbol{q}^i \;, \; r = \sum_{i=1}^{n_{\mathrm{np}}} r^i$

$$\int_{\omega_i} \left(\boldsymbol{q}^i \cdot \boldsymbol{\nabla} v + \kappa^2 r^i v\right) d\Omega = R(\phi_i v) \quad \forall v \in \mathcal{V}(\omega_i)$$

minimizing the local complementary energy

$$\int_{\omega_i} \left(\boldsymbol{q}^i \cdot \boldsymbol{q}^i + \kappa^2 (r^i)^2\right) d\Omega$$



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minimizing the global complementary energy

$$\int_\Omega \left(\boldsymbol{q} \cdot \boldsymbol{q} + \kappa^2 r^2\right) d\Omega$$

KEY POINT

Find a closed EXPLICIT solution for $\boldsymbol{q}^i$ and r^i

Local problems: $\boldsymbol{q} = \sum_{i=1}^{n_{\mathrm{np}}} \boldsymbol{q}^i$, $r = \sum_{i=1}^{n_{\mathrm{np}}} r^i$

$$\int_{\omega_i} \left(\boldsymbol{q}^i \cdot \boldsymbol{\nabla} v + \kappa^2 r^i v\right) d\Omega = R(\phi_i v) \quad \forall v \in \mathcal{V}(\omega_i)$$

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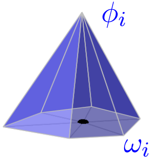

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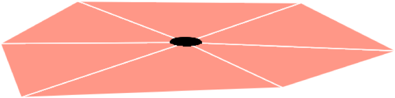
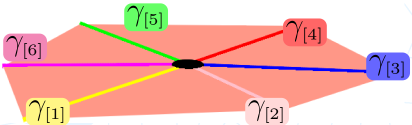


From star ω_i to elements $\Omega_k \subset \omega_i$

$$\int_{\omega_i} \left(\boldsymbol{q}^i \cdot \boldsymbol{\nabla} v + \kappa^2 r^i v \right) d\Omega = R(\phi_i v)$$

$r^i \to 0$

New guaranteed, accurate and cheap EE

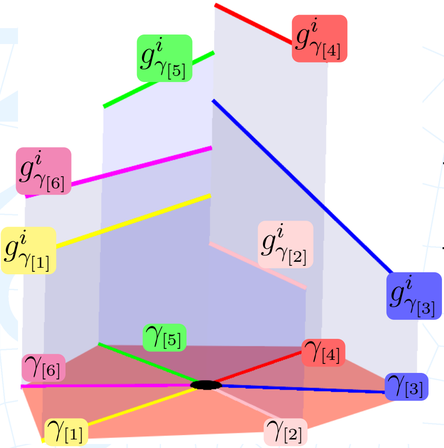


From star ω_i to elements $\Omega_k \subset \omega_i$

$$\int_{\omega_i} \boldsymbol{q}^i \cdot \boldsymbol{\nabla} v d\Omega = R(\phi_i v)$$

The explicit solution is found introducing the linear tractions on the edges of the star $\{g^i_{\gamma_{[m]}}\}$

New guaranteed, accurate and cheap EE



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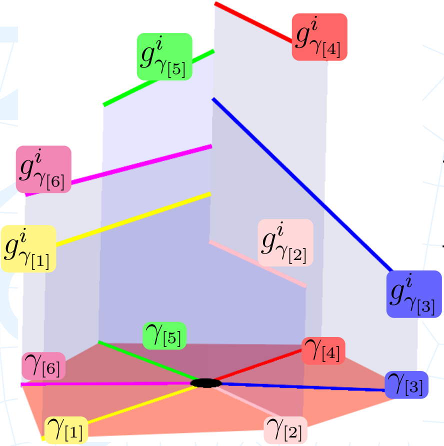

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The explicit solution is found introducing the linear tractions on the edges of the star $\{g^i_{\gamma[m]}\}$

$$\Downarrow$$

for every $\Omega_k \subset \omega_i$

Neumann BC $\quad (\boldsymbol{q}^i_k + \phi_i \boldsymbol{\nabla} u_h) \cdot \boldsymbol{n}^\gamma_k = \sigma^\gamma_k g^i_\gamma$

Divergence $\quad -\boldsymbol{\nabla} \cdot (\boldsymbol{q}^i_k + \phi_i \boldsymbol{\nabla} u_h) = \phi_i(f - \kappa^2 u_h) - \boldsymbol{\nabla} u_h \cdot \boldsymbol{\nabla} \phi_i$

New guaranteed, accurate and cheap EE

Strong form of the elementary problems:

$$-\boldsymbol{\nabla}\cdot\left(\boldsymbol{q}_k^i+\phi_i\boldsymbol{\nabla}u_h\right)=\phi_i(\ f-\kappa^2u_h\ )-\boldsymbol{\nabla}u_h\cdot\boldsymbol{\nabla}\phi_i \quad \text{in } \Omega_k$$

$$\boldsymbol{q}_k^i\cdot\boldsymbol{n}_k^\gamma=\sigma_k^\gamma g_\gamma^i-\phi_i\boldsymbol{\nabla}u_h\cdot\boldsymbol{n}_k^\gamma:=\ \mathcal{R}_{|\gamma} \quad \text{on } \partial\Omega_k$$

ASSUMPTION: for simplicity of presentation we assume that

- f is piecewise linear and
- g_{N} is piecewise constant

otherwise we need to introduce data oscillation terms

New guaranteed, accurate and cheap EE

Strong form of the elementary problems:

$$-\boldsymbol{\nabla} \cdot \left(\boldsymbol{q}_k^i + \phi_i \boldsymbol{\nabla} u_h \right) = \phi_i(\; f - \kappa^2 u_h \;) - \boldsymbol{\nabla} u_h \cdot \boldsymbol{\nabla} \phi_i \quad \text{in } \Omega_k$$

$$\boldsymbol{q}_k^i \cdot \boldsymbol{n}_k^\gamma = \sigma_k^\gamma g_\gamma^i - \phi_i \boldsymbol{\nabla} u_h \cdot \boldsymbol{n}_k^\gamma := \mathcal{R}_{|\gamma} \quad \text{on } \partial\Omega_k$$

Explicit solution: $\boxed{\boldsymbol{q}_k^i = \boldsymbol{q}_k^{iL} + \boldsymbol{q}_k^{iC}}$ as long as

$$\int_{\Omega_k} \left[\phi_i \left(f - \kappa^2 u_h \right) - \nabla u_h \cdot \nabla \phi_i \right] \, d\Omega + \sum_{\gamma \subset \partial\Omega_k} \int_\gamma \sigma_k^\gamma g_\gamma^i \, d\Gamma = 0,$$

Details can be found in

N. Parés, P. Díez, *A new equilibrated residual method improving accuracy and efficiency of flux-free error estimates*, CMAME, Volume 313, Pages 785-816 (2017)

New guaranteed, accurate and cheap EE

Strong form of the elementary problems:

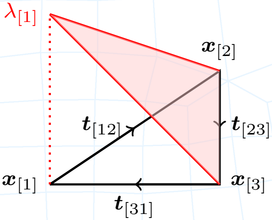
$$-\boldsymbol{\nabla}\cdot\left(\boldsymbol{q}_k^i+\phi_i\boldsymbol{\nabla}u_h\right)=\phi_i(\ f-\kappa^2u_h\ )-\boldsymbol{\nabla}u_h\cdot\boldsymbol{\nabla}\phi_i \quad \text{in } \Omega_k$$

$$\boldsymbol{q}_k^i\cdot\boldsymbol{n}_k^\gamma=\sigma_k^\gamma g_\gamma^i-\phi_i\boldsymbol{\nabla}u_h\cdot\boldsymbol{n}_k^\gamma:=\ \mathcal{R}_{|\gamma} \quad \text{on } \partial\Omega_k$$

Explicit solution of the elementary problems: $\boldsymbol{q}_k^i=\boldsymbol{q}_k^{iL}+\boldsymbol{q}_k^{iC}$

$$\boldsymbol{q}_k^{iL}=\frac{1}{2|\Omega_k|}\sum_{n=1}^{3}\sum_{\substack{m=1\\m\neq n}}\ell_{[m]}\ R|_{\gamma_{[m]}}(\boldsymbol{x}_{[n]})\ \boldsymbol{t}_{[mn]}\lambda_{[n]}$$

$$\boldsymbol{q}_k^{iC}=\frac{1}{3}\sum_{n=1}^{3}\sum_{\substack{m=2\\m>n}}^{3}\ \lambda_{[n]}\lambda_{[m]}\boldsymbol{t}_{[nm]}\boldsymbol{t}_{[nm]}^{\mathsf{T}}\ \ \nabla v^Q$$



New guaranteed, accurate and cheap EE

Strong form of the elementary problems:

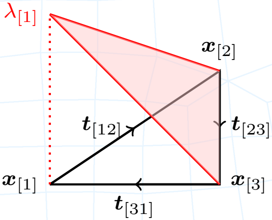
$$-\boldsymbol{\nabla}\cdot\left(\boldsymbol{q}_k^i+\phi_i\boldsymbol{\nabla}u_h\right)=\phi_i(\ f-\kappa^2u_h\ )-\boldsymbol{\nabla}u_h\cdot\boldsymbol{\nabla}\phi_i \quad \text{in } \Omega_k$$

$$\boldsymbol{q}_k^i\cdot\boldsymbol{n}_k^\gamma=\sigma_k^\gamma g_\gamma^i-\phi_i\boldsymbol{\nabla}u_h\cdot\boldsymbol{n}_k^\gamma:=\boxed{\mathcal{R}_{|\gamma}} \quad \text{on } \partial\Omega_k$$

Explicit solution of the elementary problems: $\boxed{\boldsymbol{q}_k^i=\boldsymbol{q}_k^{iL}+\boldsymbol{q}_k^{iC}}$

$$\boldsymbol{q}_k^{iL}=\frac{1}{2|\Omega_k|}\sum_{n=1}^{3}\sum_{\substack{m=1\\m\neq n}}\ell_{[m]}\boxed{R|_{\gamma_{[m]}}(\boldsymbol{x}_{[n]})}\boldsymbol{t}_{[mn]}\lambda_{[n]}$$

$$\boldsymbol{q}_k^{iC}=\frac{1}{3}\sum_{n=1}^{3}\sum_{\substack{m=2\\m>n}}^{3}\boxed{\lambda_{[n]}\lambda_{[m]}\boldsymbol{t}_{[nm]}\boldsymbol{t}_{[nm]}^{\mathsf{T}}}\ \nabla v^Q$$



$\boldsymbol{q}_k^{iL}$ imposes the tractions on the element

$\boldsymbol{q}_k^{iC}$ is traction free

New guaranteed, accurate and cheap EE

Strong form of the elementary problems:

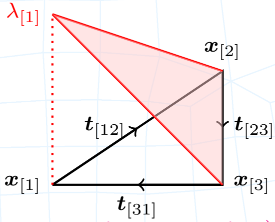
$$-\nabla \cdot \left(\boldsymbol{q}_k^i + \phi_i \nabla u_h\right) = \phi_i(\underbrace{f - \kappa^2 u_h}_{F}) - \nabla u_h \cdot \nabla \phi_i \quad \text{in } \Omega_k$$

$$\boldsymbol{q}_k^i \cdot \boldsymbol{n}_k^\gamma = \sigma_k^\gamma g_\gamma^i - \phi_i \nabla u_h \cdot \boldsymbol{n}_k^\gamma := \mathcal{R}_{|\gamma} \quad \text{on } \partial\Omega_k$$

Explicit solution of the elementary problems: $\boldsymbol{q}_k^i = \boldsymbol{q}_k^{iL} + \boldsymbol{q}_k^{iC}$

$$\boldsymbol{q}_k^{iL} = \frac{1}{2|\Omega_k|} \sum_{n=1}^{3} \sum_{\substack{m=1 \\ m\neq n}} \ell_{[m]} \; R|_{\gamma_{[m]}}(\boldsymbol{x}_{[n]}) \; \boldsymbol{t}_{[mn]} \lambda_{[n]}$$

$$\boldsymbol{q}_k^{iC} = \frac{1}{3} \sum_{n=1}^{3} \sum_{\substack{m=2 \\ m>n}}^{3} \lambda_{[n]} \lambda_{[m]} \boldsymbol{t}_{[nm]} \boldsymbol{t}_{[nm]}^{\mathsf{T}} \; \nabla v^Q$$



$$v^Q = \tfrac{3}{8}\phi_i F + \tfrac{1}{8}\left(4F_{[1]}\lambda_{[1]} - F_{[2]}\lambda_{[3]} - F_{[3]}\lambda_{[2]}\right)$$

$\boldsymbol{q}_k^{iC}$ imposes the divergence condition

New guaranteed, accurate and cheap EE

Explicit solution of the elementary problems: $\boldsymbol{q}_k^i = \boldsymbol{q}_k^{iL} + \boldsymbol{q}_k^{iC}$

$$\mathbf{q}_k^{iL} = \frac{1}{2|\Omega_k|} \sum_{n=1}^{3} \sum_{\substack{m=1 \\ m \neq n}} A_{[m]} R|_{\gamma_{[m]}}(\boldsymbol{x}_{[n]}) \boldsymbol{t}_{[mn]} \lambda_{[n]}$$

$$\mathbf{q}_k^{iC} = \frac{1}{3} \sum_{n=1}^{3} \sum_{\substack{m=2 \\ m > n}}^{3} \lambda_{[n]} \lambda_{[m]} \boldsymbol{t}_{[nm]} \boldsymbol{t}_{[nm]}^{\mathsf{T}}$$

FREE DOF g_γ^i

minimize compl. energy

subject to

$$\int_{\Omega_k} \left[\phi_i \left(f - \kappa^2 u_h\right) - \nabla u_h \cdot \nabla \phi_i\right] \, d\Omega + \sum_{\gamma \subset \partial\Omega_k} \int_\gamma \sigma_k^\gamma g_\gamma^i \, d\Gamma = 0,$$

Complementary energy $\displaystyle\int_{\Omega_k} \boldsymbol{q}_k^i(g_\gamma^i) \cdot \boldsymbol{q}_k^i(g_\gamma^i) d\Omega$

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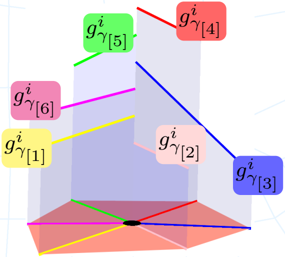
New guaranteed, accurate and cheap EE

LOCAL QUADRATIC CONSTRAINED OPTIMIZATION PROBLEM:

find $\{g^i_{\gamma_{[m]}}\}$ solution of



Minimize g^i_γ $\displaystyle\sum_{\Omega_k \subset \omega_i} \int_{\Omega_k} \boldsymbol{q}^i_k(g^i_\gamma) \cdot \boldsymbol{q}^i_k(g^i_\gamma) d\Omega$

↖ two dof per edge

Subject to $\displaystyle\int_{\Omega_k} \left[\phi_i \left(f - \kappa^2 u_h\right) - \nabla u_h \cdot \nabla \phi_i\right] \, d\Omega + \sum_{\gamma \subset \partial\Omega_k} \int_\gamma \sigma^\gamma_k g^i_\gamma \, d\Gamma = 0$

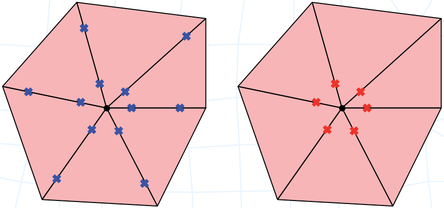
one restriction per element ↗

Hybrid-flux vs. Explicit Flux-free

Explicit Flux-free

$$\text{Minimize}_{g_\gamma^i} \quad \sum_{\Omega_k \subset \omega_i} \int_{\Omega_k} \boldsymbol{q}_k^i(g_\gamma^i) \cdot \boldsymbol{q}_k^i(g_\gamma^i) d\Omega$$

two dof per edge



one dof per edge

Hybrid-flux / equilibrated

$$\text{Minimize}_{g_\gamma} \quad g_\gamma - [[\boldsymbol{\nabla} u_h \cdot n]]_{\text{ave}}$$

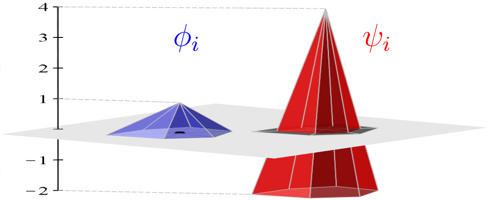
Hybrid-flux vs. Explicit Flux-free

Explicit Flux-free

$$\underset{g_\gamma^i}{\text{Minimize}} \quad \sum_{\Omega_k \subset \omega_i} \int_{\Omega_k} \boldsymbol{q}_k^i(g_\gamma^i) \cdot \boldsymbol{q}_k^i(g_\gamma^i) d\Omega$$

two dof per edge ↓

$$\text{s.t.} \quad \int_{\Omega_k} \left[\phi_i \left(f - \kappa^2 u_h\right) - \nabla u_h \cdot \nabla \phi_i\right] \; d\Omega + \sum_{\gamma \subset \partial\Omega_k} \int_\gamma \sigma_k^\gamma g_\gamma^i \; d\Gamma = 0$$



Hybrid-flux / equilibrated

$$\underset{g_\gamma}{\text{Minimize}} \quad g_\gamma - [[\boldsymbol{\nabla} u_h \cdot n]]_{\text{ave}}$$

one dof per edge ↓

$$\text{s.t.} \quad \int_{\Omega_k} \left[\psi_i \left(f - \kappa^2 u_h\right) - \nabla u_h \cdot \nabla \psi_i\right] \; d\Omega + \sum_{\gamma \subset \partial\Omega_k} \int_\gamma \sigma_k^\gamma g_\gamma \psi_i \; d\Gamma = 0$$

2D example

Uniformly forced square domain

$-\Delta u = 1$ in $[-1,1]^2$ with homogeneous Dirichlet BC

$$u(x,y) = \frac{1-x^2}{2} - \frac{16}{\pi^3} \sum_{\substack{k=1 \\ \text{odd}}}^{+\infty} \frac{\sin(k\pi(1+x)/2)(\sinh(k\pi(1+y)/2)+\sinh(k\pi(1-y)/2))}{k^3 \sinh(k\pi)}$$

$\rho = |||e|||_{ub}/|||e||| \approx 1$

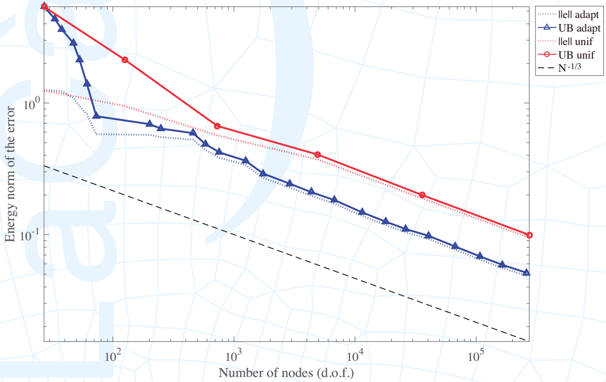
		FLUX-FREE			EQUILIBRATED
			explicit		
n_{el}	$\|\|\|e\|\|\|$	ρ^{st}	ρ	ρ^q	ρ^{eq}
8	0.34331271	1.00036	1.09131	1.01545	1.20880
32	0.27603795	1.04611	1.05288	1.03831	1.48894
128	0.15288301	1.04314	1.04621	1.03889	1.51749
512	0.07856757	1.04088	1.04470	1.03938	1.52104
2048	0.03955958	1.03948	1.04429	1.03962	1.51898
8192	0.01980831	1.03862	1.04420	1.03974	1.51641
32768	0.00990510	1.03813	1.04419	1.03982	1.51453

3D example

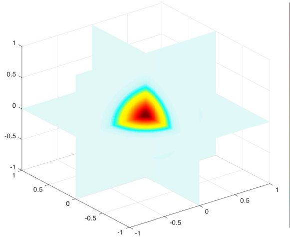
3D diffusion problem with data oscillation

$$-\Delta u = f \quad \text{in } [-1,1]^3 \quad \text{with Dirichlet BC}$$

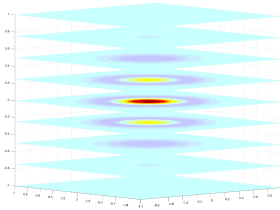
$$u(x,y,z) = e^{-20(x^2+y^2+z^2)}$$



Exact solution



Source term



3D example

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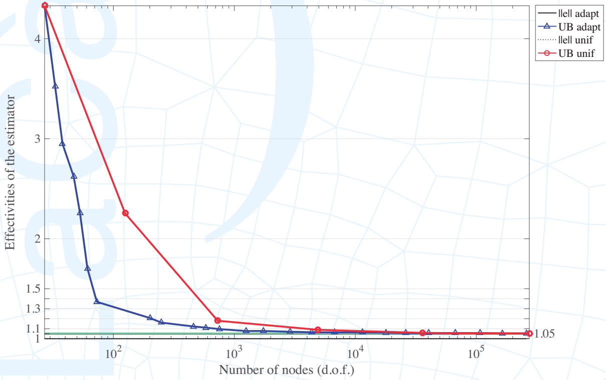
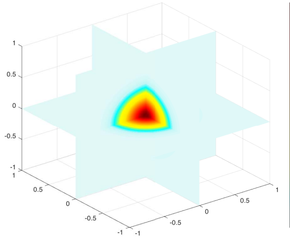
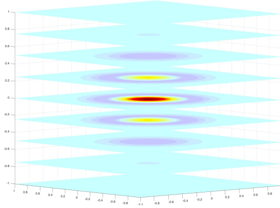


Exact solution



Source term



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3D example

3D diffusion problem with data oscillation

$$|||e|||^2 \leq \sum_{k=1}^{n_{\text{el}}} \Big(\underbrace{\|\boldsymbol{q}\|_{[\mathcal{L}^2(\Omega_k)]^3}}_{\text{dual error}} + \underbrace{\frac{h_k}{\pi}\|f - \Pi^1 f\|_{\mathcal{L}^2(\Omega_k)}}_{\text{data oscillation}} \Big)^2$$

Conclusions

- We have developed a new technique to compute guaranteed upper bounds for the energy norm of the error (which can also be used to compute bounds for QoI)
- The proposed strategy may be seen as either:
  (1) an improved *cheap* version of the flux-free estimate
  (2) a new more *efficient* hybrid-flux equilibrated EE
- Alleviating the cost of the flux-free approach does not introduce a significant difference on accuracy
- The new equilibrated tractions yield sharper bounds than the original ones

A new equilibrated residual method: improving accuracy and efficiency of flux-free error estimates in two and three dimensions

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