eXtended Hybridizable Discontinuous Galerkin (X-HDG) for Void and Bimaterial Problems

Ceren Gürkan, Esther Sala-Lardies, Martin Kronbichler, and Sonia Fernández-Méndez

Abstract A strategy for the Hybridizable Discontinuous Galerkin (HDG) solution of problems with voids, inclusions, free surfaces, and material interfaces is proposed. It is based on an eXtended Finite Element (X-FEM) philosophy with a level-set description of interfaces where the computational mesh is not required to fit the interface (i.e. the boundary). This reduces the cost of mesh generation and, in particular, avoids continuous remeshing for evolving interfaces. Differently to previous proposals for the HDG solution with unfitting meshes, the computational mesh covers the domain in our approach, avoiding extrapolations and ensuring the robustness of the method. The local problem in elements not cut by the interface and the global problem are discretized as usual in HDG. A modified local problem is considered for elements cut by the interface. At every cut element, an auxiliary trace variable on the boundary is introduced, which is eliminated afterwards using interface conditions, keeping the original unknowns and the structure of the local problem solver. The solution is enriched with Heaviside functions in case of bimaterial problems; in case of problems with voids, inclusions, or free surfaces no such enrichment is required. Numerical experiments demonstrate how X-HDG keeps the optimal convergence, superconvergence, and accuracy of HDG with no need of adapting the computational mesh to the interface boundary.

1 Introduction

Among state-of-the-art advanced discretization techniques high-order discontinuous Galerkin methods are nowadays very popular in computational fluid dynamics. DG methods inherit the advantages of finite volume methods such as stability through numerical fluxes and local conservation but they allow for the use of high-order...
approximations with a straightforward implementation of p-adaptivity. Among DG methods the hybridizable discontinuous Galerkin method (HDG) \cite{4, 5}, has been successfully applied to all kinds of problems, specifically in computational fluid dynamics (CFD), see, for instance, \cite{6, 19, 20} for its application to the Stokes and Navier–Stokes equations, or \cite{12, 15, 17} for an efficiency comparison to continuous finite elements (CFE) in the context of elliptic problems and wave propagation.

HDG inherits all the advantages of high-order DG methods \cite{2, 14, 18, 21}, such as local conservation of quantities of interest, intrinsic stabilization thanks to a proper definition of numerical fluxes at element boundaries, suitability for code vectorization and parallel computation, and straightforward realization of adaptivity. However, two main characteristics allow HDG to outperform other DG methods for problems involving self-adjoint operators, namely hybridization and superconvergence. The hybridization process allows reducing the degrees of freedom in the final linear system to the nodes in the element faces (sides in 2D), similarly to static condensation in the context of high-order continuous elements, see for instance \cite{12}. On other hand, HDG is based on a particular mixed formulation. When approximating both the primal unknown and its derivatives with polynomials of the same degree \( k \), convergence rates of order \( k + 1 \) in the \( L_2 \) norm are observed for both variables. Therefore, a simple element-by-element postprocessing of the derivatives leads to a superconvergent approximation of the primal variables, with convergence of order \( k + 2 \) in the \( L_2 \) norm. The superconvergent solution can be used to define a simple and efficient error estimator, which then can be used to drive an adaptivity process \cite{11, 13}. When it comes to the efficient solution of problems with moving boundaries and interfaces there is still work to be done. In \cite{23}, HDG has been applied to the solution of Stokes interface problems, but always considering computational meshes fitting the interface. In \cite{16}, HDG has been used for the solution of elliptic material interface problems where the material interface is defined by the computational mesh. There, the mesh needs to be adapted to properly describe the interface geometry, requiring continuous remeshing in the case of evolving interfaces. Even though a methodology for the solution of elliptic problems with meshes not fitting the boundary has been proposed in \cite{3, 8}, extrapolating the solution beyond the elements causes some restrictive requirements on the distance from the computational mesh to the boundary which as well limits the practical applicability of the proposal. An alternative strategy for the HDG solution of interface problems, based on an extended finite element (X-FEM) philosophy, is proposed here. The extended finite element method is a clever strategy to treat, for instance, the discontinuities arising at interfaces. The suitability of the X-FEM strategy for the solution of problems with moving boundaries and interfaces in the context of continuous finite elements along with high-order approximation\cite{1, 9} is well-established, see for instance, the overview paper \cite{10} and the references therein. In X-FEM, interfaces are usually represented as the zero-level set of a signed distance function, and the original computational mesh and unknown structure are maintained, with a proper numerical integration at
elements and faces cut by the interface [22]. In the case of material interfaces or cracks, the solution is enriched to represent weak or strong discontinuities across interfaces.

Motivated by the demonstrated efficiency of HDG and X-FEM, this work aims to combine these two frameworks for efficiently solving problems with moving boundaries and interfaces described by level sets. The method, labeled eXtended Hybridizable Discontinuous Galerkin (X-HDG), proposes the efficient solution of problems including voids, inclusions, free surfaces, and material interfaces. The solution is enriched with Heaviside functions in the case of material interfaces and, in the case of weak discontinuities, continuity is weakly imposed, emulating the imposition of continuity across element boundaries in standard HDG. X-HDG inherits the advantages of X-FEM methods i.e., the computational mesh is not required to adapt to the interface. This simplifies and reduces the cost of mesh generation while keeping the computational efficiency, stability, accuracy and optimal convergence of HDG. Differently to [3, 8], the computational mesh always covers the domain and, therefore, no extrapolations are required, leading to a more robust method. Hence, X-HDG keeps the superconvergence and stability properties of standard HDG, but in accordance with an X-FEM philosophy.

Section 2 presents the detailed X-HDG formulation and discretization for void problems with Neumann boundary conditions on the interface (i.e. on the boundary not fitted by the computational mesh). The local problem at elements not cut by the interface and the global problem are discretized as usual in HDG. The discretization of the local problem for cut elements is developed in Sect. 2.2. At every cut element, an auxiliary trace variable on the boundary is introduced, which is eliminated afterwards using the boundary conditions on the interface, keeping the original unknowns. The X-HDG formulation is then extended to problems with Dirichlet interfaces. The application to problems involving both Dirichlet and Neumann interfaces is straightforward. Section 3 explains the X-HDG formulation for bimaterial problems mostly referring to the slight differences in the formulation as compared to standard HDG. Here the solution space is enriched with Heaviside functions to introduce the discontinuity within the element for those elements cut by the interface. Elements not cut by the interface are treated with a standard HDG approach. Finally, numerical examples in Sect. 4 demonstrate the applicability of the method and that X-HDG keeps the accuracy, optimal convergence, and superconvergence of HDG. The application of the developed formulation to problems with evolving boundaries avoids the continuous remeshing to fit the interface, with the corresponding saving in computational time. In addition, no loss of accuracy due to the projection of quantities from one mesh to another occurs in this setting. The additional cost of X-HDG in front of HDG is mainly due to the modification of the numerical quadrature, since elemental computations for cut elements are substantially more expensive than those for standard elements. However, it is worth noting that in practical applications the ratio of cut elements to standard elements is small, and the overcost is negligible compared to the cost of remeshing.
2 X-HDG Formulation for Void Problems

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with an interior boundary $\mathcal{I}$ (also referred to as interface) and an exterior boundary $\partial\Omega^{ext} := \partial\Omega \setminus \mathcal{I}$. The following problem is considered,

\begin{align}
-\nabla \cdot (v \nabla u) &= f \text{ in } \Omega, \\
v \nabla u \cdot n &= g \text{ on } \mathcal{I}, \\
u &= u_D \text{ on } \partial\Omega^{ext},
\end{align}

where $u$ is the solution, $v$ is a material coefficient, $f$ is a given source term, $u_D$ are prescribed values on the exterior boundary, and $g$ is a prescribed flux on the interior boundary, i.e., the voids boundary. Neumann boundary conditions are considered on the interior boundary $\mathcal{I}$. The implementation of Dirichlet boundary conditions on $\mathcal{I}$ is straightforward, see Remark 2. Here, for simplicity, Dirichlet boundary conditions are considered on the exterior boundary $\partial\Omega^{ext}$; other boundary conditions at $\partial\Omega^{ext}$ do not add any difficulty, since they are implemented as in standard HDG.

The domain $\Omega$ is assumed to be covered by a finite element mesh with $n_{el}$ disjoint elements $K_i$, such that

$$\Omega \subset \bigcup_{i=1}^{n_{el}} K_i, \quad K_i \cap K_j = \emptyset \text{ for } i \neq j, \quad \partial\Omega^{ext} \subset \partial \left[ \bigcup_{i=1}^{n_{el}} K_i \right].$$

Note that the mesh fits the exterior boundary $\partial\Omega^{ext}$, but some elements may be cut by the interior boundary $\mathcal{I}$, see Fig. 1. The union of all $n_{el}$ faces $\Gamma_i$ (sides for 2D) intersecting the domain $\Omega$ is denoted by

$$\Gamma := \bigcup_{i=1}^{n_{el}} [\partial K_i \cap \Omega] = \bigcup_{j=1}^{n_{el}} [\Gamma_j \cap \Omega].$$

The discontinuous setting of (1) consists of some element-by-element equations and some global ones. The local element-by-element problems correspond to the statement of the PDE in (1) with essential boundary conditions in each element $K_i$.

Fig. 1 Two examples of domain with a void: a circular void boundary and a straight interface, $\mathcal{I}$ black.
The mesh covers the domain $\Omega$ (gray) and fits the exterior boundary $\partial\Omega^{ext} = \partial\Omega \setminus \mathcal{I}$. 

4
X-HDG for Void and Bimaterial Problems

that is,

\[ \begin{aligned}
\nabla \cdot q &= f \text{ in } K_i \\
q + \nu \nabla u &= 0 \text{ in } K_i \\
u &= \hat{u} \text{ on } \partial K_i
\end{aligned} \]

if \( \mathcal{I} \cap K_i = \emptyset \), \hspace{1cm} (2a)

\[ \begin{aligned}
\nabla \cdot q &= f \text{ in } \Omega_i \\
q + \nu \nabla u &= 0 \text{ in } \Omega_i \\
q \cdot n &= g \text{ on } \mathcal{I}_i \\
u &= \hat{u} \text{ on } \partial \Omega_i \setminus \mathcal{I}_i
\end{aligned} \]

if \( \mathcal{I} \cap K_i \neq \emptyset \) \hspace{1cm} (2b)

for \( i = 1, \ldots, n_{e1} \), where, for cut elements,

\[ \Omega_i := \Omega \cap K_i, \quad \mathcal{I}_i := \mathcal{I} \cap K_i. \] \hspace{1cm} (3)

In (2), two new variables are introduced. The quantity \( q \) represents the flux of \( u \) in order to formulate the PDE as two first order PDEs, and \( \hat{u} \) corresponds to the trace of \( u \) at the mesh faces. The local problems have been particularized for elements cut by the interior boundary (2b) and standard elements (2a).

Note that, given the trace \( \hat{u} \), which is single valued on the mesh skeleton \( \Gamma \), the local problems (2) can be solved in each element to determine the solution \( u \) and the flux \( q \). Thus, the problem now reduces to determine the trace \( \hat{u} \), with the so-called conservativity conditions (also known as global equations), that is, the continuity of the flux across element boundaries

\[ [q \cdot n] = 0 \quad \text{on } \Gamma \setminus \partial \Omega^{ext}, \] \hspace{1cm} (4)

and the boundary condition, equivalent to the exterior boundary condition in (1),

\[ \hat{u} = u_D \quad \text{on } \partial \Omega^{ext}. \] \hspace{1cm} (5)

The jump \( [\cdot] \) operator is defined at a face \( \Gamma_f \) as

\[ [\odot] = \odot_L(f) + \odot_R(f) \quad \text{on } \Gamma_f, \]

where \( R(f) \) and \( L(f) \) are numbers of the left and right elements sharing the face, that is, \( \Gamma_f = K_{L(f)} \cap K_{R(f)} \), and the subindex \( \odot_i \) denotes the value of function \( \odot \) from element \( K_i \). In particular, \( [q \cdot n] = q_{L(f)} \cdot n_{L(f)} + q_{R(f)} \cdot n_{R(f)} = (q_{L(f)} - q_{R(f)}) \cdot n_{L(f)}. \)

It is important noting that the continuity of the solution \( u \) across \( \Gamma \) is imposed by the Dirichlet boundary condition in the local problems (2) and the fact that \( \hat{u} \) is single valued on \( \Gamma \).

The discretization of the conservativity condition (4) and the local problems (2), with the boundary condition (5), leads to the X-HDG formulation.
The next sections present the details of the X-HDG formulation, stating the discretization of the local problems for standard and cut elements and the discretization of the conservativity condition (4). The local problem at elements not cut by the interface (2a) and the global problem are discretized as usual in HDG [4, 5], as recalled in Sects. 2.1 and 2.3. The discretization of the local problem for cut elements (2b) is developed in Sect. 2.2.

To simplify the presentation, the same notation is used for the numerical approximation, belonging to the usual finite dimensional spaces of piecewise polynomials on elements, and the exact solution, that is \( u, q \) and \( \hat{u} \).

### 2.1 Local Problem for Standard Elements

This section recalls the standard HDG local problem on an element \( K_i \) not cut by the interface. It corresponds to the discretization of (2a), that is: Given \( \hat{u} \in \mathcal{A}^h \), find \( u \in \mathcal{P}_k(K_i), q \in [\mathcal{P}_k(K_i)]^d \) where \( \mathcal{P}_k \) denotes the space of polynomials of degree less or equal to \( k \), such that

\[
\begin{align*}
\int_{K_i} v \nabla \cdot q \, dV + \int_{\partial K_i} \tau v (u - \hat{u}) \, dS &= \int_{K_i} v f \, dV \quad \forall v \in \mathcal{P}_k(K_i), \\
\int_{K_i} q \cdot w \, dV - \int_{K_i} v u \nabla \cdot w \, dV + \int_{\partial K_i} v \hat{u} w \cdot n \, dS &= 0 \quad \forall w \in [\mathcal{P}_k(K_i)]^d
\end{align*}
\]

The first equation in (6) can be derived from the first equation in (2a) by applying integration by parts, replacing the flux by the numerical flux

\[
\hat{q} := q + \tau (u - \hat{u}) n,
\]

and undoing the integration by parts. The second equation is obtained from the weak form of the second equation in (2a), applying integration by parts and replacing the boundary condition \( u = \hat{u} \) on the element boundary.

**Remark 1** The parameter \( \tau \), appearing in the definition of the numerical flux (7), is a non-negative stabilization parameter usually taken of order \( O(\nu) \). For each element, it may be taken as a positive constant on all faces, or positive on one arbitrary face and zero on the rest (single face). Both options lead to stable and optimally convergent solutions, with superconvergent post-processed solutions. See for instance [4, 11] for details on the influence of this parameter on the solution behavior.

The discretization of the local problem (6) leads to a system of equations of the form

\[
\begin{align*}
A^K_{uu} u^i + A^K_{uq} q^i + A^K_{ui} \Lambda^i &= f^K_i, \\
A^K_{qu} u^i + A^K_{qq} q^i + A^K_{qi} \Lambda^i &= 0,
\end{align*}
\]

(8)
where \( \mathbf{u}^i \) and \( \mathbf{q}^i \) are the vectors of nodal values of \( u \) and \( q \) in element \( K_i \), and \( \mathbf{A}^i \) is the vector of nodal values of \( \hat{u} \) on the \( n \) faces of the element (\( n = 3 \) for triangles and \( n = 4 \) for tetrahedra). That is,

\[
\mathbf{A}^i := \begin{bmatrix}
\hat{\mathbf{u}}^{F_1} \\
\vdots \\
\hat{\mathbf{u}}^{F_m}
\end{bmatrix}
\]  

(9)

where \( \hat{\mathbf{u}}^i \) denotes the nodal values of \( \hat{u} \) on face \( F_j \), and \( F_{ij} \) is the number of the \( j \)-th face of element \( K_i \). Note that the subindices in the \( \mathbf{A} \) matrices refer to the space for the weighting function and the test function, respectively.

System (8) can be solved for \( \mathbf{u}^i \) and \( \mathbf{q}^i \) in each element, obtaining the so-called \textit{local solver} in the element \( K_i \)

\[
\mathbf{u}^i = \mathbf{U}^{K_i} \mathbf{A}^i + \mathbf{f}_U^{K_i}, \quad \mathbf{q}^i = \mathbf{Q}^{K_i} \mathbf{A}^i + \mathbf{f}_Q^{K_i},
\]

(10)

with

\[
\begin{bmatrix}
\mathbf{U}^{K_i} \\
\mathbf{Q}^{K_i}
\end{bmatrix} = -\mathbf{A}^{-1} \begin{bmatrix}
\mathbf{A}^{K_i}_{uu} & \mathbf{A}^{K_i}_{uq} \\
\mathbf{A}^{K_i}_{qu} & \mathbf{A}^{K_i}_{qq}
\end{bmatrix}, \quad \begin{bmatrix}
\mathbf{f}_U^{K_i} \\
\mathbf{f}_Q^{K_i}
\end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix}
\mathbf{f}_U^k \\
\mathbf{0}
\end{bmatrix}
\]

(11)

and

\[
\mathbf{A} = \begin{bmatrix}
\mathbf{A}^{K_i}_{uu} & \mathbf{A}^{K_i}_{uq} \\
\mathbf{A}^{K_i}_{qu} & \mathbf{A}^{K_i}_{qq}
\end{bmatrix}.
\]

That is, for each element, the elemental values of the solution, \( \mathbf{u}^i \) and \( \mathbf{q}^i \), can be explicitly expressed in terms of the trace on its faces, \( \mathbf{A}^i \).

### 2.2 Local Problem for a Cut Element

The X-HDG local problem at an element \( K_i \) cut by the interior boundary corresponds to the discretization of (2b), that is: Given \( \hat{\mathbf{u}} \in \mathbb{A}^h \), find \( \mathbf{u} \in \mathcal{P}_k(\Omega_i) \), \( \mathbf{q} \in [\mathcal{P}_k(\Omega_i)]^d \) such that

\[
\begin{align*}
\int_{\Omega_i} \mathbf{v} \nabla \cdot \mathbf{q} \, dV + \int_{\partial \Omega_i \setminus \Omega_i} \mathbf{v} \tau (u - \hat{u}) \, dS + \int_{\Omega_i} \tau v (u - \hat{u}) \, dV + \int_{\Omega_i} \mathbf{v} f \, dV \\
\int_{\Omega_i} \mathbf{q} \cdot \mathbf{w} \, dV - \int_{\Omega_i} \mathbf{v} u \nabla \cdot \mathbf{w} \, dV + \int_{\partial \Omega_i \setminus \Omega_i} v \mathbf{u} \tau \cdot \mathbf{n} \, dS + \int_{\Omega_i} 
\end{align*}
\]

(12)
for all \( v \in \mathcal{P}_k(\Omega_i) \) and \( w \in [\mathcal{P}_k(\Omega_i)]^d \), where \( \Omega_i = \Omega \cap K_i \), and \( \tilde{u}^i \) is a new trace variable approximating the trace of the solution on the interface \( \mathcal{I}_i = \mathcal{I} \cap K_i \), see Fig. 2. Compared to the weak form for standard elements (6), the X-HDG weak form for a cut element has two additional terms corresponding to integrals along the interface \( \mathcal{I}_i \), involving the new trace variable \( \tilde{u}^i \).

The discretization of the local problem (12) leads to the system of equations

\[
\begin{align*}
\left[ A_{uu}^{\Omega} + A_{u\tilde{u}}^{\Omega} \right] \tilde{u}^i + A_{uq}^{\Omega} q^i + A_{u\tilde{u}}^{\Omega} \tilde{u}^i + A_{u\tilde{u}}^{\mathcal{I}} \tilde{u}^i &= f_i^{\Omega}, \\
A_{q\tilde{u}}^{q} \tilde{u}^i + A_{q\tilde{q}}^{\Omega} q^i + A_{q\tilde{u}}^{\mathcal{I}} \tilde{u}^i + A_{q\tilde{u}}^{\mathcal{I}} \tilde{u}^i &= 0,
\end{align*}
\]

(13)
similar to (8), but with three new matrices corresponding to integrals on the Neumann boundary (marked with the superindex \( I_i \)), and the nodal values for the new trace variable, \( \tilde{u}^i \). The local problem is closed by imposing the Neumann boundary condition on \( \mathcal{I}_i \), i.e.,

\[
\hat{q} \cdot n = g \quad \text{on} \quad \mathcal{I}_i.
\]

Using the expression of the numerical flux \( \hat{q} \) defined in (7), i.e. \( \hat{q} \cdot n = q \cdot n + \tau (u - \tilde{u}^i) \), the weak form of the Neumann condition in \( \mathcal{I}_i \) is to find \( \tilde{u}^i \in \mathcal{P}_k(\mathcal{I}_i) \) such that

\[
\int_{\mathcal{I}_i} \tilde{v} q \cdot n \ dS + \tau \int_{\mathcal{I}_i} \tilde{v} (u - \tilde{u}^i) \ dS = \int_{\mathcal{I}_i} \tilde{v} g \ dS \quad \forall \tilde{v} \in \mathcal{P}_k(\mathcal{I}_i).
\]

(14)

The discretization of the Neumann boundary condition leads to a discrete equation of the form

\[
A_{u\tilde{u}}^{\mathcal{I}} \tilde{u}^i + A_{u\tilde{q}}^{\mathcal{I}} q^i + A_{u\tilde{u}}^{\mathcal{I}} \tilde{u}^i = g,
\]

that allows expressing the new trace values \( \tilde{u}^i \) in terms of elemental values

\[
\tilde{u}^i = T_{u}^{i} u^i + T_{q}^{i} q^i - t^i
\]

(15)
X-HDG for Void and Bimaterial Problems

with

\[ T^i_u = -\left[ A^\nu_{u' u''} \right]^{-1} A^\nu_{u' u''}, \quad T^i_q = -\left[ A^\nu_{q' q''} \right]^{-1} A^\nu_{q' q''}, \quad t^i = -\left[ A^\nu_{u' q''} \right]^{-1} g. \]

Replacing (15) in (13) leads to the final discrete local problem

\[ \begin{bmatrix} A^\Omega_{uu} + A^\nu_{uu} + A^\nu_{u' u''} T^i_u \\ A^\Omega_{q'u} + A^\nu_{q'u} + A^\nu_{q'u' q''} T^i_q \end{bmatrix} \begin{bmatrix} u^i \\ q^i \end{bmatrix} + \begin{bmatrix} A^\Omega_{uq} + A^\nu_{uq} T^i_u \\ A^\Omega_{q'q'} + A^\nu_{q'q'} T^i_q \end{bmatrix} \begin{bmatrix} q^i \\ A^\nu_{q'q'} A^i \end{bmatrix} = \begin{bmatrix} f^\Omega_u + A^\nu_{u' u''} t^i \\ A^\nu_{q'q'} t^i \end{bmatrix} \]

Now, similarly to (8), this system can be solved for \( u^i \) and \( q^i \), obtaining the local solver in the cut element \( K_i \), i.e., Eq. (10) with

\[ \begin{bmatrix} U^{K_i} \\ Q^{K_i} \end{bmatrix} = -A^{-1} \begin{bmatrix} A^\Omega_{uq} \\ A^\nu_{q'q'} \end{bmatrix} \begin{bmatrix} f^K_{u} \\ f^K_{q} \end{bmatrix} = A^{-1} \begin{bmatrix} f^\Omega_u + A^\nu_{u' u''} t^i \\ A^\nu_{q'q'} t^i \end{bmatrix} \]

and

\[ A = \begin{bmatrix} A^\Omega_{uu} + A^\nu_{uu} + A^\nu_{u' u''} T^i_u \\ A^\Omega_{q'u} + A^\nu_{q'u} + A^\nu_{q'u' q''} T^i_q \end{bmatrix} \begin{bmatrix} A^\Omega_{uq} + A^\nu_{uq} T^i_u \\ A^\Omega_{q'q'} + A^\nu_{q'q'} T^i_q \end{bmatrix} \]

Note that the structure of the local solver is exactly the same as for non-cut elements (10), as the internal trace variable \( \tilde{u}^i \) has been isolated and it is not an unknown of the problem anymore.

Remark 2 In case of Dirichlet boundary conditions imposed over the interface \( \mathcal{I} \), the X-HDG formulation is straightforward. Once the system in (13) is obtained, the terms \( A^\nu_{uq} \tilde{u}^i \) and \( A^\nu_{q'u} \tilde{u}^i \) are already known so they can simply be moved to the right hand side of the system to obtain a local solver that looks like:

\[ \begin{bmatrix} A^\Omega_{uu} + A^\nu_{uu} \\ A^\Omega_{q'u} + A^\nu_{q'u} \end{bmatrix} u^i + A^\Omega_{uq} q^i + A^\nu_{u' u''} A^i = f^\Omega_u + f^\Omega_{u' u''} \]

where \( A^\nu_{u' u''} u^i = f^\nu_{u'} \) and \( A^\nu_{q'u} u^i = f^\nu_{q'u} \). Analogous to the Neumann case, (17) can be solved for \( u^i \) and \( q^i \) to obtain the local solver for each element and finally the problem can be closed by solving global equation defined in detail in Sect. 2.3. Note that interface problems where some interfaces are subject to Neumann conditions and others are subject to Dirichlet conditions can be naturally covered by selecting the appropriate strategy element-by-element.
2.3 Global Problem

The local problem, both in a standard element or in a cut element, leads to the local solver (10), that computes the solution in the element, $u$ and $q$, in terms of the trace values at its boundary, $\hat{u}$. Thus, now the problem is reduced to determine the trace nodal values $\{\hat{u}^f\}_{f=1}^{N_{te}}$ on the mesh skeleton $\Gamma$. For this purpose the so-called global problem is stated, which corresponds to the discretization of the conservativity condition on $\Gamma$ (4).

Replacing $q$ by the numerical flux (7), the weak form for the trace variable is:

\[
\text{Find } \hat{u} \in \Lambda^h \text{ such that } \int_{\Gamma} \hat{v} [q \cdot n] dS + 2\tau \int_{\Gamma} \hat{v} (\{u\} - \hat{u}) dS = 0 \quad \forall \hat{v} \in \Lambda^h,
\]

where $\{\cdot\}$ is the mean operator on the faces,

\[
\{\circ\} = \frac{1}{2} \left( \circ_{L(f)} + \circ_{R(f)} \right) \quad \text{on } \Gamma_f.
\]

As usual in HDG, the discretization of this equation for every face $\Gamma_f$ leads to an equation of the form

\[
A_{fiu}^{fL} u^{L(f)} + A_{fiq}^{fL} q^{L(f)} + A_{fR}^{fiu} u^{R(f)}(f) + A_{fR}^{fiq} q^{R(f)}(f) + A_{fiu}^{fR} \hat{u}^f = 0.
\] (18)

Replacing the local solver (10), for the elements $K_{L(f)}$ and $K_{R(f)}$, in (18) for every face $\Gamma_f$, leads to a system of equations involving only the trace variables $\{\hat{u}^f\}_{f=1}^{N_{te}}$.

As usual in an HDG code, the implementation of the method involves a loop over elements. For each element, the matrices and vectors for the local solver (10) are computed, and the contribution to the Eq. (18) is assembled for each face of the element. Once the system is assembled for all elements, and Dirichlet boundary conditions (5) are imposed, the linear system can be solved. Then, given the trace variables $\{\hat{u}^f\}_{f=1}^{N_{te}}$, the interior values $u^i$ and $q^i$, can be computed for each element using (10). Note that X-HDG keeps the structure of a standard HDG code. The only difference is the modified local problem on cut elements (13), and the corresponding matrices in the local solver (16).

Remark 3 An element-by-element postprocessing procedure allows to compute an X-HDG superconvergent solution. Similarly to standard HDG the superconvergent solution can be computed in every element $K_i$ as the solution of: Find $u^* \in \mathcal{P}_{k+1}(\Omega_i)$ such that

\[
\int_{\Omega_i} v \nabla u^* \cdot \nabla v \, dV = - \int_{\Omega_i} q \cdot \nabla v \, dV \quad \forall v \in \mathcal{P}_{k+1}(\Omega_i),
\]

\[
\int_{\Omega_i} u^* \, dV = \int_{\Omega_i} u \, dV,
\]
with $\Omega_i = K_i$ for standard elements, and $\Omega_i = K_i \cap \Omega$ for cut elements. The solution of this element-by-element computation, $u^*$, converges with order $k + 2$ in the $L_2$ norm. See [4, 7] for details and other possible computations of a superconvergent solution.

3 X-HDG Formulation for Bimaterial Problems

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain divided into two disjoint subdomains, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, with an interface $\mathcal{I} = \Omega_1 \cap \Omega_2$. The following bimaterial problem is considered,

$$
\begin{aligned}
-\nabla \cdot (\nu \nabla u) &= f \text{ in } \Omega_1 \cup \Omega_2, \\
[u_n] &= 0 \text{ on } \mathcal{I}, \\
[v \nabla u \cdot n] &= 0 \text{ on } \mathcal{I}, \\
u &= u_D \text{ on } \partial \Omega,
\end{aligned}
$$

(19)

where $u$ is the solution, $\nu$ is a material coefficient with discontinuous definition across the interface (that is, $\nu = \nu_i$ in $\Omega_i$ for $i = 1, 2$), $f$ is a given source term and $u_D$ are prescribed values at the exterior boundary. The jump $[\cdot]$ operator is defined at an interface (material interface or, later, faces between elements), using values from the domains to the left and right of the interface, $[\odot] = \odot_1 + \odot_2$, always involving the normal vector on the interface.

The partition of $\Omega$ in elements $K_i$ induces a new problem equivalent to (19), with some element-by-element equations and some global ones. The local problem at uncut element is the one of standard HDG i.e., equivalent to Eq. (2a), and its discretization can be found in Sect. 2.1. For elements cut by the interface the local problem is

$$
\begin{aligned}
\nabla \cdot q &= f \text{ in } K_i \setminus \mathcal{I} \\
q + \nu \nabla u &= 0 \text{ in } K_i \setminus \mathcal{I} \\
[u_n] &= 0 \text{ on } \mathcal{I} \cap K_i \text{ if } \mathcal{I} \cap K_i \neq \emptyset, \\
[q \cdot n] &= 0 \text{ on } \mathcal{I} \cap K_i \\
u &= \hat{u} \text{ on } \partial K_i
\end{aligned}
$$

(20)

and its X-HDG discretization is stated in Sect. 3.1.

The problem is closed with the conservativity condition (4), corresponding to the imposition of continuity across element boundaries, and the boundary condition $\hat{u} = u_D$ on $\partial \Omega$. Its weak form is the same as for standard HDG, see Sect. 2.1.
The following discrete spaces for elemental variables, \( u \) and \( q \), and for trace variables, \( \hat{u} \), are considered,

\[
\mathcal{V}^h := \{ v \in L^2(\Omega) : v|_{K_i} \in \mathcal{P}_k(K_i) \quad \text{if} \quad K_i \cap I = \emptyset, \\
v|_{K_i} \in \mathcal{P}_k(K_i) \oplus H\mathcal{P}_k(K_i) \quad \text{if} \quad K_i \cap I \neq \emptyset \},
\]

\[
\mathcal{A}^h := \{ \hat{v} \in L^2(\Gamma) : \hat{v}|_{\Gamma_i} \in \mathcal{P}_k(\Gamma_i) \quad \text{if} \quad \Gamma_i \cap I = \emptyset, \\
\hat{v}|_{\Gamma_i} \in \mathcal{P}_k(\Gamma_i) \oplus H\mathcal{P}_k(\Gamma_i) \quad \text{if} \quad \Gamma_i \cap I \neq \emptyset \},
\]  

(21)

where \( H \) is a Heaviside function that enriches the approximation in cut elements and on cut faces. The Heaviside function can be defined, for instance, as

\[
H = \begin{cases} 
1 & \text{in } \Omega_1 \\
-1 & \text{in } \Omega_2 
\end{cases}
\]

to introduce discontinuities across the interface \( I \).

In every cut element \( K_i \), given the considered enriched space \( \mathcal{P}_k(K_i) \oplus H\mathcal{P}_k(K_i) \), \( u \) is assumed to be discontinuous across the interface \( I \). The continuity of the solution and the normal flux across the interface is then imposed as usual in HDG by means of an auxiliary trace variable on the interface, which is eliminated afterwards keeping the original unknowns.

At cut faces, a discontinuous approximation for the trace \( \hat{u} \) is also considered but, in this case, there is no need to impose weak continuity on it. In this way the resulting formulation is equivalent to a standard HDG formulation on a mesh where some elements are split (for instance into a triangle and a quadrilateral) to fit the interface, with the usual face-by-face discontinuous approximation of the trace.

This definition ensures that the proposed X-HDG formulation keeps the super-convergence and stability properties of standard HDG accordance with an X-FEM philosophy using the original computational mesh with an enriched approximation at cut elements and faces.

### 3.1 Local Problem at a Bimaterial Element

The X-HDG local problem at an element \( K_i \) cut by the interface corresponds to the discretization of (20), that is: Given \( \hat{u} \in \mathcal{A}^h \), find \( u \in \mathcal{P}_k(K_i) \oplus H\mathcal{P}_k(K_i) \), \( q \in [\mathcal{P}_k(K_i) \oplus H\mathcal{P}_k(K_i)]^d \) such that

\[
\int_{K_i \setminus I_i} v \nabla \cdot q \, dV + \int_{\partial K_i} \tau v (u - \hat{u}) \, dS + 2 \int_{I_i} \tau \{ v v (u - \hat{u}) \} \, dS = \int_{K_i \setminus I_i} uf \, dV
\]

\[
\int_{K_i \setminus I_i} \frac{1}{2} q \cdot w \, dV - \int_{K_i \setminus I_i} u \nabla \cdot w \, dV + \int_{\partial K_i} \hat{w} \cdot v \, dS + \int_{I_i} \hat{u} [ w \cdot n ] \, dS = 0
\]  

(22)
for all \( v \in \mathcal{P}_k(K_i) \oplus H^1\mathcal{P}_k(K_i) \) and \( w \in [\mathcal{P}_k(K_i) \oplus H^1\mathcal{P}_k(K_i)]^d \), where \( I_i := K_i \cap I \) is the interface segment in the element and \( \tilde{u}^i \) is a new trace variable approximating the trace of the solution on the interface \( I_i \). The mean operator \( \{ \cdot \} \) is defined taking values from the left and right domains sharing the interface or the mesh face, \( \{ \otimes \} = \frac{1}{2} (\otimes_1 + \otimes_2) \).

The X-HDG local problem (22) can be derived following the standard HDG rationale for each one of the domains, \( K_i \cap \Omega_1 \) and \( K_i \cap \Omega_2 \), and summing the obtained weak forms. Compared to the weak form for standard elements (6), the X-HDG weak form for a cut element has two additional terms corresponding to integrals along the interface \( I_i \) involving the new trace variable \( \tilde{u}^i \) and taking into account the discontinuous nature of the approximation space. These new terms weakly impose the condition \( u = \tilde{u}^i \) on \( I_i \), ensuring the weak imposition of continuity of \( u \) across the interface.

The local problem is now closed by imposing the conservativity condition across the interface \( I_i \), \( \llbracket q \cdot n \rrbracket = 0 \) on \( I_i \). The weak form of this conservativity condition in \( I_i \) can be written as: given \( u \in \mathcal{P}_k(K_i) \oplus H^1\mathcal{P}_k(K_i) \), \( q \in [\mathcal{P}_k(K_i) \oplus H^1\mathcal{P}_k(K_i)]^d \), find \( \tilde{u}^i \in \mathcal{P}_k(I_i) \) such that

\[
\int_{I_i} \tilde{v}\llbracket q \cdot n \rrbracket \, dS + 2\tau \int_{I_i} \tilde{v} \left( \{ vu \} - \{ v \tilde{u}^i \} \right) \, dS = 0 \quad \forall \tilde{v} \in \mathcal{P}_k(I_i).
\]

Similarly to the void cut element with Neumann boundary conditions, Eq. (23) can be used to isolate \( \tilde{u}^i \) in terms of \( u^i \) and \( q^i \). Then, replacing in (22) a local problem with the same structure as standard HDG (10), but modified matrices, is obtained.

### 4 Numerical Tests

The performance of the novel X-HDG method is tested on three numerical examples. The Laplace equation with known analytical solution is solved over a square domain with a circular void in Sect. 4.1. Boundary condition on the circular boundary is of Neumann type in the first case and of Dirichlet type in the second. The accuracy and the convergence of X-HDG for void problems is tested and compared to HDG with a mesh adapted to the void boundary.

Section 4.2 demonstrates the performance of X-HDG for bimaterial problems where the domain is divided by a straight interface. On each side of the interface different materials are considered and the corresponding error analysis is done. As usual, the performance of X-HDG is compared with the one of HDG by solving exactly the same problem with HDG settings, i.e., a mesh fitting to the interface.

In all numerical tests, the stabilization parameter is \( \tau = 1 \) on all faces.
4.1 X-HDG with Voids

The first numerical example assesses the performance of X-HDG with Neumann boundary conditions imposed at the void interface. The Laplace Eq. (19), with \( \nu = 1 \), is solved over a square domain with a centered circular void with radius 0.41, \( \Omega = (-1, 1)^2 \setminus B((0, 0), 0.41) \). Neumann boundary conditions are imposed on the void boundary \( \mathcal{I} = \partial B((0, 0), 0.41) \), and Dirichlet boundary conditions are imposed on the exterior boundary \( \partial \Omega \setminus \mathcal{I} = \partial ((-1, 1)^2) \). Dirichlet and Neumann values and the source term \( f \) are set so that the analytical solution is

\[
u(x, y) = \exp \left( 0.1 \sin(5.1x - 6.2y) + 0.3 \cos(4.3x + 3.4y) \right)\].

Figure 3 shows the computational domain and the analytical solution.

Figure 4 presents the computational mesh for X-HDG and for standard HDG, respectively. For the X-HDG computation, a regular triangular mesh in the square domain \((-1, 1)^2\) is considered, covering the domain \(\Omega\) and fitting the exterior boundary \(\partial \Omega \setminus \mathcal{I}\). A level set function is used to describe the boundary of the void, \(\mathcal{I}\). Three kinds of elements appear in the computational mesh. Elements inside the domain (dark gray) are treated as standard HDG elements. For elements cut by the interior boundary \(\mathcal{I}\) (light gray), the modified X-HDG local problem is considered, see Sect. 2.2. The elements that are totally inside the void (white) have no contribution to the solution, so they are simply disregarded. The computational mesh for standard HDG is adapted to fit the void boundary, with similar uniform mesh size.

Figure 5 shows the evolution of the \(L_2\) error for decreasing uniform mesh size, for X-HDG and HDG with degrees \(k = 2, 3, 4\), for both the solution and the postprocessed superconvergent solution, see Remark 3. X-HDG keeps optimal convergence with rates close to \(k+1\) for the solution, and \(k+2\) for the postprocessed solution, with similar levels of accuracy as HDG, while getting rid of adapting the mesh to the void boundary.

![Fig. 3 Circular void example: domain \(\Omega\) and void boundary \(\mathcal{I}\), and analytical solution](image)
X-HDG for Void and Bimaterial Problems

Fig. 4 Circular void example: computational mesh for X-HDG and for standard HDG. The X-HDG mesh is not adapted to the void boundary. Elements in the interior of the domain are colored in dark gray. Elements in softer gray are elements cut by the interface \( \mathcal{I} \). Elements in white are inside the void, and are not considered in the computation.

Fig. 5 Neumann circular void example: convergence plots for X-HDG (left) and HDG (right). The numbers correspond to the slope of each segment, and they are underlined for postprocessed solution.

Remark 4 The description of the interface is done using a level set function given by its nodal values on the \( k \)-degree mesh used for the solution, and piece-wise \( k + 1 \) degree level set function is used for the computation of post-processed solution to ensure \( k + 2 \) convergence rate reached by the postprocessed solution. For standard HDG this is equivalent to requiring a boundary approximation of polynomial degree \( k + 1 \) properly fitting the description of the boundary for the superconvergent postprocessing.

Now, Dirichlet boundary conditions imposed over the interface \( \mathcal{I} \). The domain, the boundary conditions on the exterior boundary, and source term definitions are
The numbers correspond to the slope of each segment, and they are underlined for postprocessed solution kept same as before. Convergence plots are shown in Fig. 6. Again, a one to one resemblance is observed between the X-HDG and HDG results, keeping optimal convergence properties of HDG with an unfitted computational mesh.

4.2 Bimaterial X-HDG

Finally, the Laplace equation (19) over a square domain divided into two disjoint subdomains by a linear interface is solved in order to demonstrate the efficiency of X-HDG formulation for bimaterial problems. The size of our domain is $(-1, 1)^2$ and the interface $I$ is located at $x = 0.2031$ with the viscosity parameter being $\nu_1 = 1$ in the left side of the interface i.e., at $\Omega_1$ and being $\nu_2 = 2.5$ in $\Omega_2$ as shown in Fig. 7.

Homogenous Neumann boundary conditions are set on the interface $I$ and homogenous Dirichlet boundary conditions set on the outer boundary $\partial \Omega \setminus I = \partial ((-1, 1)^2)$ where the analytical solution reads

$$u(x) = \begin{cases} 5x^5 & \text{in } \Omega_1, \\ 2x^5 + A & \text{in } \Omega_2, \end{cases}$$

with jump $A = 3(0.2031)^5$. In order to solve this problem, the procedure explained in Sect. 3 is followed for elements cut by the linear interface. The standard HDG formulation is used for elements not cut by the interface.

Figure 8 shows the convergence history of X-HDG and HDG for the bimaterial interface problem. Starting with an initial mesh with four elements per coordinate direction, we have done four mesh refinements—each refinement doubling the element number on the sides—both for X-HDG and HDG with varying approximation.
X-HDG for Void and Bimaterial Problems

Fig. 7 Bimaterial X-HDG: computational mesh for X-HDG and for standard HDG after two mesh refinements. On the left, the X-HDG mesh not adapted to the linear interface boundary $I$ which shown in black. Elements in white are standard elements in domain one whereas elements in dark gray are standard elements in domain two. With light gray, elements cut by the interface $I$ are shown. On the right, HDG mesh fitting to the linear interface is shown. Elements in domain one are shown in white whereas elements in domain two shown in light gray.

Fig. 8 Bimaterial X-HDG: Convergence history of X-HDG and HDG. Slopes for the solution on each segment are shown in bold, slopes for post-processed solution are shown in bold and underlined.

degree between $k = 1$ to $k = 4$. The results verify the X-HDG formulation for bimaterial interface problems when compared to HDG. Over the fine mesh and for higher approximation degree we can see the slight effect of matrix ill-conditioning in both X-HDG and HDG convergence history but this does not prevent any of the methods to reach optimal $k + 1$ and $k + 2$ (super-) convergence rates.
5 Conclusions

A strategy for the HDG solution of boundary value problems with voids, inclusions, free boundaries, and material interfaces has been proposed. It is based on an extended finite element (X-FEM) philosophy with a level-set description of interfaces. With this new strategy, the computational mesh does not need to be adapted to the interface (i.e.; the boundary), simplifying and reducing the cost of mesh generation. In case of evolving interfaces, the cost of continuous re-meshing, and the approximation errors due to the projection from one mesh to another, are avoided.

Differently to previous attempts for the HDG solution with meshes that do not fit the boundary, the computational mesh covers the domain in our method, avoiding extrapolations and ensuring the robustness of the method. The local problem in elements not cut by the interface and the global problem are discretized as usual in HDG. A modified local problem is considered for elements cut by the interface. In every cut element, an auxiliary trace variable on the boundary is introduced, which is eliminated element-by-element using the boundary conditions on the interface, keeping the original unknowns and the structure of the local problem solver. In case of bimaterial problems the solution space is enriched with Heaviside functions to introduce the discontinuity within the element for the elements cut by the interface whereas elements not cut by the interface are treated with standard HDG approach. Although the special treatment for cut elements due to the modified numerical integration causes a slight increase in CPU time, in practical applications the extra cost becomes negligible compared to the cost of mesh adaptation or re-meshing.

The proposed method has been compared to standard HDG on three numerical tests: the solution of a Laplace problem with known analytical solution in a square domain with a circular void with Dirichlet and Neumann boundary conditions, and the solution of the Laplace equation over a square domain with two different materials being divided by a linear interface. Standard HDG computations have been done on a mesh fitting the voids boundaries and material interfaces for comparison with X-HDG that uses a regular mesh covering the domain. The numerical tests assert that X-HDG keeps the HDG optimal convergence rates for the solution, the gradient, and the post-processed super-convergent solution, without the need to adapt the mesh to the boundary. In all tests, similar accuracy is observed for a similar mesh size.

Acknowledgements This work was supported by the DAFOH2 project (Ministerio de Economía y Competitividad, MTM2013-46313-R), and the Erasmus Mundus Joint Doctorate SEED project (European Commission).
X-HDG for Void and Bimaterial Problems

References
